Counting independent sets in hypergraphs and its applications

József Balogh
U. of Illinois at U.C.

2015
Ralph Faudree (1939 – 2015)
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- 50 joint papers with Erdős.
Erdős, Faudree and Sós conjectured that every $c$-Ramsey graph with $n$ vertices contains $\Omega(n^5/2)$ induced subgraphs, any two of which differ either in the number of vertices or in the number of edges, i.e., the number of distinct pairs $(|V(H)|, |E(H)|)$, as $H$ ranges over all induced subgraphs of $G$, is $\Omega(n^5/2)$.

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Laci Székely 60!
Plenary Speaker:
Plenary Speaker:

József Balogh:

Extremal results for random discrete structures
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József Balogh:

Extremal results for random discrete structures

Will talk about recent results and current standing of things!
Balogh–Morris–Samotij, Saxton–Thomason [2015]

Certain hypergraphs have only few independent sets.
Authors I. [at the time of the submission of the paper]

W. Samotij

R. Morris
Authors II. [at the time of the submission of the paper]
Authors I. [at the time of the acceptance of the paper]
Authors II. [at the time of the acceptance of the paper]

Recent Results:
Wildcats -- Red Alliance 3 -- 2 (1-1) [Goals: Samu 2, Aiden]

Current Standing:
1. Wildcats 11 1 - 42 - 9 34 [already champion!]
Counting Independent sets in Hypergraphs

Balogh–Morris–Samotij, Saxton–Thomason [2015]

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Example (Turán problem)

- \( V = \) edges of \( K_n \),
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- Independent sets in $\mathcal{H} < \quad \quad \quad \quad \quad \quad >$ $K_k$-free graphs on $[n]$.
- There are only few $K_k$-free graphs on $[n]$. 
The number of triangle-free graphs: 
Regularity Lemma approach

Theorem (Erdős–Kleitman–Rothschild [1976])

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$. 
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$$O(1) \cdot n^n \cdot 2^{n^2/4} \cdot \binom{n^2}{o(n^2)} = 2^{n^2/4+o(n^2)}.$$
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There is a $t < 2^{O(\log n \cdot n^{3/2})}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is an $i \in [t]$ such that $H \subseteq G_i$. 
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- For each $F_n$ triangle-free graph there is an $i$ that $F_n \subset G_i$.
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- For each \(F_n\) triangle-free graph there is an \(i\) that \(F_n \subseteq G_i\).
- \(e(G_i) \leq n^2/4 + o(n^2)\).
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- For each $F_n$ triangle-free graph there is an $i$ that $F_n \subset G_i$.
- $e(G_i) \leq n^2/4 + o(n^2)$.
- Number of choices for $F_n$ is $t \cdot 2^{n^2/4 + o(n^2)} = 2^{n^2/4 + o(n^2)}$. 
Szemerédi Container Lemma's

**Szemerédi container lemma**

There is a $t = 2^{o(n^2)}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is $i \in [t]$ such that $H \subseteq G_i$. 
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There is a $t = 2^{o(n^2)}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is $i \in [t]$ such that $H \subseteq G_i$.

Szemerédi Approximate container lemma

There is a $t = O(1)$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each triangle-free, such that for every triangle-free graph $H$ there is a permutation of the vertex set, that there is an $i \in [t]$ such that $|E(H) - E(G_i)| = o(n^2)$. 
Let $f : E(K_n) \to [0, 2]$ be uniform, random function, independent for each edge.

Random points in the metric polytope

The polytope contains the cube $[1, 2]^{(n^2)}$.

What is the volume of this polytope? $1 \leq \ldots \leq 2(n^2)$.

It is $(1 + o(1))(n^2)$ with the Regularity Lemma.

It is between $n^{3/2}$ and $n^{9/5} + o(1)$ with the entropy upper bound.
Metric Spaces

Kozma, Meyerovitch, Peled and Samotij [2013+++]:
Random points in the metric polytope

- Let $f : E(K_n) \to [0, 2]$ be uniform, random function, independent for each edge.
- $f$ is metric if every triangle satisfies the triangle inequality:

$$\forall u, v, w \in V(K_n) : \quad f(uv) + f(uw) \geq f(vw).$$

- Metric functions form a polytope in $\mathbb{R}^{n^2}$.
- This polytope contains the cube $[1, 2]^{n^2}$.
- What is the volume of this polytope? $1 \leq \cdots \leq 2(n^2)$.
- It is $(1 + o(1))n^2$ [Using Regularity Lemma!].
- It is between $n^{3/2}$ and $n^{9/5} + o(1)$ [Upper bound using entropy].
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- Metric functions form a polytope in $\mathcal{R}(\binom{n}{2})$. 
Kozma, Meyerovitch, Peled and Samotij [2013+++]:
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- Metric functions form a polytope in \( \mathcal{R}^{\binom{n}{2}} \).
- This polytope contains the cube \( [1, 2]^{\binom{n}{2}} \).
Kozma, Meyerovitch, Peled and Samotij [2013+++]:
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**Using Regularity Lemma!**

**Upper bound using entropy.**
Let $f : E(K_n) \to [0, 2]$ be uniform, random function, independent for each edge.

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Metric functions form a polytope in $\mathcal{R}^{\binom{n}{2}}$.

This polytope contains the cube $[1, 2]^{\binom{n}{2}}$.

What is the volume of this polytope? $1 \leq \ldots \leq 2^{\binom{n}{2}}$.

It is $(1 + o(1))^{\binom{n}{2}}$ [Using Regularity Lemma!].
Let $f : E(K_n) \to [0, 2]$ be uniform, random function, independent for each edge.

$f$ is metric if every triangle satisfies the triangle inequality:

$$\forall u, v, w \in V(K_n) : \quad f(uv) + f(uw) \geq f(vw).$$

Metric functions form a polytope in $\mathcal{R}^{n \choose 2}$.

This polytope contains the cube $[1, 2]^{n \choose 2}$.

What is the volume of this polytope? $[1 \leq \ldots \leq 2^{n \choose 2}]$.

It is $(1 + o(1))^{n \choose 2}$ [Using Regularity Lemma!].

It is between $n^{3/2}$ and $n^{9/5 + o(1)}$ [Upper bound using entropy].
Discretize the problem: $f : E(K_n) \to [2r]$. 

Consider functions as $2^r$-edge-colored $K_n$. 

Apply 'Colored Regularity Lemma'. 

Cluster graph edges maybe multicolored! 

Cluster graph is 'metric'. 

Product of multiplicities of each triangle is $\leq (r+1)^3$. 

Number of metric edge-colorings is at most $(r+1)(n^2) + o(n^2)$.

Mubayi, Terry (2015+): Almost every metric function is $f : E(K_n) \to [r, \ldots, 2r]$. 
Kozma, Meyerovitch, Peled and Samotij: Regularity Lemma Approach

- Discretize the problem: \( f : E(K_n) \rightarrow [2r] \).
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- Cluster graph edges maybe multicolored!
- Cluster graph is ‘metric’.
- Product of multiplicities of each triangle is \( \leq (r + 1)^3 \).
- Number of metric edge-colorings is at most \( (r + 1)\binom{n}{2} + o(n^2) \).
Discretize the problem: $f : E(K_n) \rightarrow [2r]$.

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Cluster graph is ‘metric’.

Product of multiplicities of each triangle is $\leq (r + 1)^3$.

Number of metric edge-colorings is at most $(r + 1)^{n\choose 2} + o(n^2)$.

$f : E(K_n) \rightarrow [r, \ldots, 2r]$ is metric, bound ‘sharp’.
Discretize the problem: \( f : E(K_n) \rightarrow [2r] \).

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\( f : E(K_n) \rightarrow [r, \ldots, 2r] \) is metric, bound ‘sharp’.

Almost every metric function is \( f : E(K_n) \rightarrow [r, \ldots, 2r] \).
Balogh, Wagner [2015+]: Container Lemma Approach

\[ V(\mathcal{H}) := E(K_n) \times [2r]. \]
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- $V(\mathcal{H}) := E(K_n) \times [2r]$.
- $E(\mathcal{H}) := \{(uv, i), (vw, j), (uw, \ell) \mid (i, j, \ell) \text{ is a non-metric triple}\}$. 
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- Metric independent set: \( E(K_n) \times [r + 1, \ldots, 2r] \).
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- Metric independent set: \( E(K_n) \times [r + 1, \ldots, 2r] \).
- Supersaturation:
  
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  \text{if } S \subset V(\mathcal{H}), |S| > (1 + c)(r + 1)\binom{n}{2} \text{ then } e(\mathcal{H}[S]) > 0.1c\binom{n}{3}.
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Balogh, Wagner [2015+]: Container Lemma Approach

- $V(\mathcal{H}) := E(K_n) \times [2r]$.
- $E(\mathcal{H}) := \{[(uv, i), (vw, j), (uw, \ell)] : (i, j, \ell) \text{ is a non-metric triple}\}$.
- Independent transversal sets are metric colorings!!
- Large independent set:
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Balogh, Wagner [2015+]: Container Lemma Approach

Number of metric edge-colorings is $(r + 1)\binom{n}{2} + o(n^2)$, when $r \leq n^{1/3-o(1)}$. 

Balogh, Wagner [2015+]: Container Lemma Approach

The volume of the metric polytope is at most $n^{11/6+o(1)}$. 
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Kozma, Meyerovitch, Morris, Peled and Samotij: Container Approach

The volume of the metric polytope is between $n^{3/2}$ and $n^{3/2} \log^2 n$. 
Extremal Graph Theory in random graphs:

Babai, Simonovits, Spencer [1990]

(i) For what $p$ it is true that the largest triangle-free subgraph of $G(n, p)$ is bipartite?
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Expected number of $C_4$-free graphs in $G(n, p)$ with at least $m$ edges is $o(1)$ for $p < 1/16$. 
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Assuming that $H$ contains a cycle, the number of $H$-free graphs is

$$2^{(1+o(1))\text{ex}(n,H)}.$$
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- Fix an extremal graph on $n/2$ vertices.
- Replace each vertex with 2 vertices.
- Replace each edge with a matching. Generate many $C_6$-free graphs:

$$7^{\text{ex}(n/2,C_6)} > 2^{\text{ex}(n,C_6)}.$$
$C_4$-free graphs in random graphs:

$p \cdot \text{ex}(n, C_4) \leq \text{ex}(G(n, p), C_4) \leq \text{ex}(n, C_4)$.
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Balogh, Wagner [2015+]

For $p < 1/5$ there is a $c > 0$ that $(1 + c)p \cdot \text{ex}(n, C_4) \leq \text{ex}(G(n, p), C_4)$ w.h.p.
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For every \( p \in (0, 1) \), there is a \( c > 0 \) that the largest \( C_4 \)-free subgraph of \( G(n, p) \) has at most \((1 - c) \cdot \text{ex}(n, C_4)\) edges w.h.p.
The number of $C_4$-free graphs is $2^{O(ex(n,C_4))} = 2^{O(n^{3/2})}$.
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The Kleitman-Winston Method

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- $F_i := H_i^2$, $V(F_i) = V(H_i)$, $E(F_i) = \{uv : d_{H_i}(u, v) = 2\}$. 

For a vertex ordering, degree sequence, a (small) container containing its neighborhood for each vertex.
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Want to embed a $C_4$-free graph $H$ with $(1 - c) \cdot \text{ex}(n, C_4)$ edges.
C₄-free graphs in random graphs:

Balogh, Wagner [2015+]

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- $C_i$ is not much larger than $d_i$!
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- For a fixed vertex ordering, degree sequence, containers: can corresponding $H$ be in $G(n, p)$?
- $E(H)$ does not have many spaces to be placed, random graph likely will not contain there sufficient amount of edges.
General framework — examples

Example (Erdős–Turán problem)

- \( V = \{1, \ldots, n\} \),
- \( \mathcal{H} = k \)-term APs in \([n]\).
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- $V = \text{edges of } K_n$,
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Example (sum-free sets)
- $V = \text{an Abelian group}$,
- $\mathcal{H} = \text{sets of the form } \{x, y, z\} \text{ with } x + y = z$ (Schur triples).
Theorem (Balogh–Morris–Samotij [2015])

For every $k, c, \varepsilon$ there is a $C$ that the following holds. Let $\mathcal{H} \subseteq \binom{V}{k}$ such that for $\ell \in [k]$, $p \in [0, 1]$

$$\Delta_\ell(\mathcal{H}) \leq c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$
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Let $\mathcal{F} = \{A \subseteq V : |\mathcal{H}[A]| \geq \varepsilon \cdot e(\mathcal{H})\}$. Similar result was obtained independently by Saxton and Thomason.

Explain: Example of triangle-free graphs.
Theorem (Balogh–Morris–Samotij [2015])

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- a very small family $S \subseteq \binom{V(\mathcal{H})}{\leq Cp \cdot v(\mathcal{H})}$ of labels,
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**Theorem (Balogh–Morris–Samotij [2015])**

For every \( k, c, \varepsilon \) there is a \( C \) that the following holds. Let \( \mathcal{H} \subseteq \binom{V}{k} \) such that for \( \ell \in [k], p \in [0, 1] \)

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Let \( \mathcal{F} = \{ A \subseteq V : |\mathcal{H}[A]| \geq \varepsilon \cdot e(\mathcal{H}) \} \). Then there are:

- a very small family \( S \subseteq \binom{V(\mathcal{H})}{\leq Cp \cdot v(\mathcal{H})} \) of labels,
- \( f : S \to \mathcal{F}^c \) (maps each label to a set that is sparse in \( \mathcal{H} \)),
- a labeling function \( g : I(\mathcal{H}) \to S \),

Similar result was obtained independently by Saxton and Thomason.

Explain: Example of triangle-free graphs.
Transference Theorem

**Theorem (Balogh–Morris–Samotij [2015])**

For every $k, c, \varepsilon$ there is a $C$ that the following holds. Let $\mathcal{H} \subseteq \binom{V}{k}$ such that for $\ell \in [k], \ p \in [0, 1]$

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Explain: Example of triangle-free graphs.
Transference Theorem: — illustration

- Dense sets
- Independent sets
- Small sets (labels)
Transference Theorem: — illustration

\[ \mathcal{F} \]

.dense sets

\[ f(S) \]

.covering sets

\[ \mathcal{I}(\mathcal{H}) \]

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Transference Theorem: illustration

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\[ g \]

\[ g(l) \]
Transference Theorem: — illustration

- dense sets
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\[ f(g(I)) \]

\[ f(S) \]
Transference Theorem: — illustration

Let \( f(g(I)) \) be the function representing the transference of dense sets from \( I(\mathcal{H}) \) to \( \mathcal{F} \), where \( I(\mathcal{H}) \) represents independent sets and \( \mathcal{F} \) represents covering sets. The function \( f(S) \) indicates the mapping from covering sets to the image set.

The small sets (labels) are denoted by \( g(I) \).
Transference theorems

Theorem (Conlon–Gowers [2009+], Schacht [2009+])

extremal result $\mathcal{R}$

$\Rightarrow$ random analogue of $\mathcal{R}$.

supersaturation

Dr D. Conlon  Sir W.T. Gowers  Dr M. Schacht
Szemerédi’s theorem

Theorem (Szemerédi [1975])

For every $k \geq 3$, the largest subset of $\{1, \ldots, n\}$ with no $k$-term AP has $o(n)$ elements.

Endre Szemerédi
Random analogue of Szemerédi’s theorem

Theorem (Kohayakawa–Łuczak–Rödl [1996])

For every $\delta > 0$, there exists a $C$ such that if $p(n) \geq Cn^{-1/2}$, then a.a.s.: the $p$-random subset $[n_p]$ satisfies:

Every $A \subseteq [n]_p$ with $|A| \geq \delta|[n]_p|$ contains a 3-term AP.
Transference theorems — corollary

**Theorem (Conlon–Gowers [2009+], Schacht [2009+])**

extremal result $\mathcal{R}$

$+$

$\implies$

random analogue of $\mathcal{R}$

supersaturation

For every $k \geq 3$ and $\delta > 0$, if $p(n) \geq C(k, \delta) \cdot n^{-1/k-1}$, then a.a.s. $\mathcal{A}$ satisfies that every $A \subseteq [n]_p$ with $|A| \geq \delta|\mathcal{A}|$ contains a $k$-term AP.
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$+ \quad \implies \quad$ random analogue of $\mathcal{R}$

supersaturation

**Corollary (Random analogue of Szemerédi’s theorem)**

For every $k \geq 3$ and $\delta > 0$, if $p(n) \geq C(k, \delta) \cdot n^{-\frac{1}{k-1}}$, then a.a.s. $[n]_p$ satisfies that every $A \subseteq [n]_p$ with $|A| \geq \delta|[n]_p|$ contains a $k$-term AP.
Theorem (Turán [1941])

For every $k \geq 3$,

$$
\text{ex}(n, K_k) = e(T_{k-1}(n)) = \left(1 - \frac{1}{k-1} + o(1)\right) \binom{n}{2}.
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### Theorem (Conlon–Gowers [2009+], Schacht [2009+])

For $p = p(n) \gg n^{-\frac{2}{k+1}}$ a.a.s.:
Transference theorems — corollary

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**Theorem (Conlon–Gowers [2009+], Schacht [2009+])**

For $p = p(n) \gg n^{-\frac{2}{k+1}}$ a.a.s.:

$$\text{ex}(G(n, p), K_k) = \left(1 - \frac{1}{k-1} + o(1)\right) \cdot e(G(n, p)).$$

This is usually referred to as the random analogue of Turán’s theorem.
Certain hypergraphs have only few independent sets.
Counting Independent sets in Hypergraphs

Balogh–Morris–Samotij, Saxton–Thomason [2015]

Certain hypergraphs have only few independent sets.

Corollary (Random analogue of Szemerédi’s theorem)

For every $k \geq 3$ and $\delta > 0$, if $p(n) \geq C(k, \delta) \cdot n^{-\frac{1}{k-1}}$, then a.a.s. $[n]_p$ satisfies that every $A \subseteq [n]_p$ with $|A| \geq \delta|\binom{n}{2}|$ contains a $k$-term AP.
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Theorem (Erdős–Kleitman–Rothschild [1976])

There are at most $2^{(1+o(1)) \cdot \text{ex}(n,K_k)} K_k$-free graphs on $n$ vertices.