The Local Action Lemma

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28th Cumberland Conference
2015
Lovász Local Lemma

Let $\mathcal{A}$ be a finite set of “bad” random events in a probability space $\Omega$.

For $A \in \mathcal{A}$ let $\Gamma(A)$ be a subset of $\mathcal{A} \setminus \{A\}$ such that $A$ is independent from the $\sigma$-algebra generated by $\mathcal{A} \setminus (\Gamma(A) \cup \{A\})$.

**Theorem (Lovász & Erdős 1975)**

Suppose that there exists an assignment of reals $\mu : \mathcal{A} \rightarrow [0; 1)$ such that for every $A \in \mathcal{A}$ we have

$$\Pr(A) \leq \mu(A) \prod_{B \in \Gamma(A)} (1 - \mu(B)).$$

Then $\Pr(\bigcap_{A \in \mathcal{A}} \overline{A}) > 0$. In fact, $\Pr(\bigcap_{A \in \mathcal{A}} \overline{A}) \geq \prod_{A \in \mathcal{A}} (1 - \mu(A))$. 

A sequence $w = a_1a_2\ldots a_n$ is *nonrepetitive* if it does not contain a repetition: $w \neq xyyz$ (for $y \neq \emptyset$).
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Example: nonrepetitive sequences

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Theorem (Thue 1906)
For every $n$ there exists a nonrepetitive sequence of 0s, 1s, and 2s of length $n$.

List version: Given a sequence $L_1, L_2, \ldots, L_n$ of sets, is there a nonrepetitive sequence $a_1a_2\ldots a_n$ with each $a_i \in L_i$?
Example: nonrepetitive sequences

Conjecture
For any sequence $L_1, L_2, \ldots, L_n$ of sets such that $|L_i| \geq 3$ there is a nonrepetitive sequence $a_1a_2\ldots a_n$ with each $a_i \in L_i$.

Applying the LLL gives the bound $|L_i| \geq 64$. 
A coloring \( f : V \to C \) of a graph \( G(V, E) \) is nonrepetitive if for every path \( P = v_1 v_2 \ldots v_\ell \) in \( G \) the sequence \( f(v_1)f(v_2)\ldots f(v_\ell) \) is nonrepetitive.

**Figure:** Bad coloring.
A coloring $f : V \rightarrow C$ of a graph $G(V, E)$ is nonrepetitive if for every path $P = v_1 v_2 \ldots v_\ell$ in $G$ the sequence $f(v_1)f(v_2)\ldots f(v_\ell)$ is nonrepetitive.

**Figure:** Good coloring.
A coloring $f : V \to C$ of a graph $G(V, E)$ is nonrepetitive if for every path $P = v_1v_2\ldots v_\ell$ in $G$ the sequence $f(v_1)f(v_2)\ldots f(v_\ell)$ is nonrepetitive.

Figure: Bad coloring—a nonrepetitive coloring must be proper.
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\[ c = 2e^{16} \text{ (Alon et al. 2002).} \]
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**Theorem (Alon, Grytczuk, Hałuszczak, & Riordan 2002)**

$$\pi(G) \leq c\Delta(G)^2.$$  

$c = 12.92$ (Haranta & Jendrol’ 2012).
Effective version?

Observation: $\Pr \left( \bigcap_{A \in \mathcal{A}} \overline{A} \right)$ can be exponentially small.

Under the conditions of the LLL, is there an effective (randomized) algorithm that finds a point $\omega \in \Omega$ avoiding all the “bad” events $\mathcal{A}$?
The answer is YES! (Moser & Tardos 2010)

To analyse the algorithm, Moser & Tardos invented the so-called entropy compression method.

It turns out that we can apply this method directly (without using the LLL), and get better results!
Example: nonrepetitive sequences

Conjecture

For any sequence $L_1, L_2, \ldots, L_n$ of sets such that $|L_i| \geq 3$ there is a nonrepetitive sequence $a_1a_2\ldots a_n$ with each $a_i \in L_i$. 
Example: nonrepetitive sequences

Theorem (Grytczuk, Kozik, & Micek 2013)
For any sequence $L_1, L_2, \ldots, L_n$ of sets such that $|L_i| \geq 4$ there is a nonrepetitive sequence $a_1a_2\ldots a_n$ with each $a_i \in L_i$. 
Example: nonrepetitive colorings

Theorem (Haranta & Jendrol’ 2012)

\[ \pi(G) \leq 12.92 \Delta(G)^2. \]
Example: nonrepetitive colorings

Theorem (Dujmović, Joret, Kozik, & Wood 2013)

\[ \pi(G) \leq (1 + o(1))\Delta(G)^2. \]
Other examples

The entropy compression method has been successfully applied to acyclic colorings and acyclic edge colorings of graphs, star colorings, facial nonrepetitive colorings of planar graphs, etc.
General framework?

The Main Question

Is there a general probabilistic statement (analogous to the LLL) that implies combinatorial results obtained using the entropy compression method?
The idea

- $\mathcal{X}$—the set of all possible “configurations”;
- $\mathcal{G} \subseteq \mathcal{X}$—the set of all “good” configurations;
- $X \in \mathcal{X}$—a random variable;
- **WANT**: $\Pr(X \in \mathcal{G}) > 0$. 

$B$—a set of “local” functions on $\mathcal{X}$;
$M \supseteq B$—all the compositions of functions from $B$. 

$\mathcal{G}$ is closed under the functions in $M$. 

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$\mathcal{G}$ is closed under the functions in $B$: if $x \in \mathcal{G}$, $\beta \in B$, then $\beta(x) \in \mathcal{G}$. 
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Example

- $\mathcal{X}$—the set of all *partial* colorings of a graph;
- $\mathcal{G} \subseteq \mathcal{X}$—the set of all “good” colorings;
- $X \in \mathcal{X}$—a random *total* coloring;
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- $X \in \mathcal{X}$—a random total coloring;
- WANT: $\Pr(X \in \mathcal{G}) > 0$ (i.e. a randomly chosen total coloring is “good”).

- $B$ contains the functions $f_v$ for each vertex $v$; the function $f_v$ uncolors $v$;
- $M \supseteq B$ contains the functions $f_S$ for each set of vertices $S$; the function $f_S$ uncolors all the vertices in $S$. 
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- $M \supseteq B$ contains the functions $f_S$ for each set of vertices $S$; the function $f_S$ uncolors all the vertices in $S$.

- $\mathcal{G}$ is closed under the functions in $B$: if a coloring is “good”, it stays good after uncoloring some of the vertices.
If $\gamma \in M$, $\beta \in B$, then

$$\Pr(\gamma(X) \in G) \leq \Pr(\beta \gamma(X) \in G).$$

We would like to have a bound in the other direction:

$$\zeta(\beta) \Pr(\gamma(X) \in G) \geq \Pr(\beta \gamma(X) \in G)$$

for some $\zeta : B \to [1, +\infty)$. The LAL provides conditions on a function $\zeta : B \to [1, +\infty)$ which guarantee that (1) holds.
Bounding the probability

If $\gamma \in M$, $\beta \in B$, then

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The LAL provides conditions on a function \( \zeta : B \to [1; \infty) \) which guarantee that (1) holds.
Local Action Lemma: the statement

**Theorem (B.)**

For every $x \in \mathcal{X} \setminus \mathcal{G}$ and $\beta \in B$ such that $\beta(x) \in \mathcal{G}$ choose an arbitrary element $g_\beta(x) \in M$. For $\beta \in B$ and $\alpha \in M$ define

$$P(\beta, \alpha) := \sup_{\gamma \in M} \Pr (g_\beta(\gamma(X)) = \alpha \mid \alpha \beta \gamma(X) \in \mathcal{G}).$$

Suppose that for some $\alpha_0 \in M$ we have $\Pr(\alpha_0(X) \in \mathcal{G}) > 0$. If there exists a function $\zeta : B \to [1; +\infty)$ such that for every $\beta \in B$ we have

$$\zeta(\beta) \geq 1 + \sum_{\alpha \in M} P(\beta, \alpha) \zeta(\alpha \beta),$$

then $\Pr(X \in \mathcal{G}) > 0$. In fact,

$$\zeta(\alpha_0) \Pr(X \in \mathcal{G}) \geq \Pr(\alpha_0(X) \in \mathcal{G}).$$
Concluding remarks

- The LAL implies all the new combinatorial results obtained using the entropy compression method.
- It also implies the LLL (even the Lopsided LLL).
- The LAL gives an explicit lower bound for the probability (which implies the bound given by the LLL).
- The LAL does not put any restrictions on the distribution (it does not have to be uniform or independent).
Thank you!