Coloring Squares of Planar Graphs with no 4-cycles and no 5-cycles

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- The chromatic number of \( G \), denoted \( \chi(G) \), is the smallest value \( k \) such that \( G \) has a proper \( k \)-coloring.
Coloring Planar Graphs

- Four Color Theorem (1976): If $G$ is a planar graph, then $\chi(G) \leq 4$. 

Grötzsch's Theorem (1959): If $G$ is a planar graph without 3-cycles, then $\chi(G) \leq 3$.

Conjecture (Steinberg, 1976): If $G$ is a planar graph without 4-cycles or 5-cycles, then $\chi(G) \leq 3$. 

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Def: If $G$ is a graph, then $G^2$ is the graph with the same vertex set as $G$, and where $u$ and $v$ are adjacent in $G^2$ if and only if $dist_G(u, v) \in \{1, 2\}$.

Finding a proper vertex coloring of $G^2$ is equivalent to finding a coloring of $G$ where vertices cannot have the same color as each other if they are within distance 2 of each other.

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Conjecture (Wegner, 1977): Let $G$ be a planar graph with maximum degree $\Delta$. Then $\chi(G^2) \leq \begin{cases} 
7 & \text{if } \Delta = 3 \\
\Delta + 5 & \text{if } 4 \leq \Delta \leq 7 \\
\left\lfloor \frac{3\Delta}{2} \right\rfloor + 1 & \text{otherwise}
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To use substantially fewer than $\frac{3\Delta}{2}$ colors, we must forbid 4-cycles.

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Dvořák et al. (2007): If $G$ is a planar graph of girth at least six and with $\Delta \geq 30$, then $\chi(G^2) \leq \Delta + 2$. 

Provided constructions of graphs of girth six and arbitrarily large $\Delta$ needing $\Delta + 2$ colors.

Zhu, Lu, Wang, and Chen (2012): If $G$ is a planar graph without 4-cycles or 5-cycles and with $\Delta \geq 9$, then $\chi(G^2) \leq \Delta + 5$.

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Elements of the graph are given a value called "charge". The charge is moved around (never created or destroyed) according to specially tailored "discharging rules". By assuming certain structures do not appear in the graph, we can reach some contradiction based on the charge. Thus the graph must contain one of the given structures. Often the structures chosen are those that cannot appear if the graph is a minimal counterexample to some claim. We call such structures reducible for the claim.

If a counterexample \(G\) cannot contain some configuration, but by discharging it must contain it, then no such counterexample exists.
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Now we redistribute charge to get a final charge $ch^*$, and show that $ch^*(x) \geq 0$ for all $x \in V \cup F$. This gives the following contradiction:

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We use this to show that certain structures cannot appear in $G$. 

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Hence the minimum degree of any vertex in $G$ is 2, and if $G$ contains a 2-vertex $u$ on a triangle, then the sum of the degrees of the neighbors of $u$ is at least $\Delta + 5$. 
Let $N^2(u)$ denote the 2-neighborhood of $u$, i.e. the set of all vertices at distance at most 2 from vertex $u$. Not hard to see that $|N^2(u)| \leq \sum_{v \in N(u)} d(v)$.

Main Reducibility Lemma: Two adjacent vertices $u$ and $v$ such that $|N^2(u)| \leq \Delta + 3$ and $|N^2(v)| \leq \Delta + 2$ are reducible.

Example application: adjacent 2-vertices are reducible (each has a 2-neighborhood of size at most $\Delta + 2$). With these forbidden structures in mind, we now move on to the discharging phase.
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Rule 1: Faces Give Charge

- Recall initial charges: $ch(f) = \ell(f) - 4$. 

Note: after $R1$, all faces have $ch^*(f) \geq 0$. 

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Coloring Squares of Planar Graphs with no 4- or 5-cycles
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Rule 2: High-Degree Vertices Give Charge

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\[
\begin{align*}
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d(v_i) < 5 \text{ for } 1 \leq i \leq 5:
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![Diagram showing vertex v1 and its neighbors v2, v3, v4, v5, with charges distributed evenly among neighbors]
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$d(v_1) \geq 5$:
Rule 2: High-Degree Vertices Give Charge

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\[
d(v_1) \geq 5:
\]

If \( v \) is a 5\(^+\)-vertex, then its excess initial charge is \( d(v) - 4 \). If all its neighbors have lower degree, then it gives them each \( d(v) - 4 \) charge.

As \( d(v) \to \infty \), the amount \( v \) gives to each neighbor approaches 1.
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If \( v \) is a 5\(^+\)-vertex, then its excess initial charge is \( d(v) - 4 \). If all its neighbors have lower degree, then it gives them each \( \frac{d(v) - 4}{d(v)} \).
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\[ d(v_1) \geq 5: \]

- If \( v \) is a 5\(^+\)-vertex, then its excess initial charge is \( d(v) - 4 \). If all its neighbors have lower degree, then it gives them each \( \frac{d(v) - 4}{d(v)} \).
- As \( d(v) \to \infty \), the amount \( v \) gives to each neighbor approaches 1.
Rules 3 and 4

- **R3:** If a $4^+$-vertex $u$ is on a 3-face $uvw$, where $w$ is either (i) a 2-vertex or (ii) a 3-vertex with a 2-neighbor, then $u$ passes the charge $c$ on to $w$.

![Diagram](image)
Rules 3 and 4

- **R3**: If a $4^+$-vertex $u$ is on a 3-face $uvw$, where $w$ is either (i) a 2-vertex or (ii) a 3-vertex with a 2-neighbor, and $u$ receives some charge $c$ during R2 from $v$,

\begin{itemize}
  \item[(i)]
  \begin{align*}
  w & \quad v \\
  \quad u & \quad C
  \end{align*}
  \\
  \begin{itemize}
    \item[(ii)]
    \begin{align*}
    w & \quad v \\
    \quad u & \quad C
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**Rules 3 and 4**

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- **R4:** If a 3-vertex has excess (positive) charge after R1-R3, it splits this charge evenly among its neighbors with negative charge.
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- **R4:** If a $3^+$-vertex has excess (positive) charge after R1-R3, it splits this charge evenly among its neighbors with negative charge.
Recall: initial charges sum to \(-8\). To reach a contradiction, we need to show that all elements end with nonnegative final charge.
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- We already saw that all faces have nonnegative charge after R1.
- Each $4^+$-vertex starts with nonnegative initial charge and never gives away more than its “positive balance”, thus ends with nonnegative final charge.
Recall: initial charges sum to $-8$. To reach a contradiction, we need to show that all elements end with nonnegative final charge.

We already saw that all faces have nonnegative charge after $R1$.

Each $4^+$-vertex starts with nonnegative initial charge and never gives away more than its "positive balance", thus ends with nonnegative final charge.

The bulk of the work is therefore in analyzing the $2$-vertices and $3$-vertices.
3-vertices

- A 3-vertex $u$ has initial charge $= 3 - 4 = -1$. 

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3-vertices

- A 3-vertex $u$ has initial charge $= 3 - 4 = -1$.
- If $u$ is not on a triangle, then all needed charge comes from incident edges.
3-vertices

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![Diagram of three vertices with charges labeled 1/6 each]
3-vertices

- A 3-vertex $u$ has initial charge $= 3 - 4 = -1$.
- If $u$ is not on a triangle, then all needed charge comes from incident edges.

If $u$ is on a triangle, let $v_1$ and $v_2$ denote the neighbors on the triangle and $w$ denote the neighbor off the triangle.
3-vertices

- A 3-vertex $u$ has initial charge $= 3 - 4 = -1$.
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If $u$ is on a triangle, let $v_1$ and $v_2$ denote the neighbors on the triangle and $w$ denote the neighbor off the triangle.

- If $d(w) \geq 6$, then $u$ gets all needed charge from $uw$ and $w$. 
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- If $d(w) \geq 6$, then $u$ gets all needed charge from $uw$ and $w$.
- If $d(v_1) \geq 12$, then $u$ gets all needed charge from $uw$ and $v_1$. 
3-vertices

- A 3-vertex $u$ has initial charge $= 3 - 4 = -1$.
- If $u$ is not on a triangle, then all needed charge comes from incident edges.

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- If $d(v_1) \geq 12$, then $u$ gets all needed charge from $uw$ and $v_1$.
- (Similarly, if $d(v_1) + d(v_2) \geq 16$, then $u$ gets enough charge).
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- A 3-vertex \( u \) has initial charge \( = 3 - 4 = -1 \).
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\[ \begin{array}{c}
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
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\end{array} \]

- If \( u \) is on a triangle, let \( v_1 \) and \( v_2 \) denote the neighbors on the triangle and \( w \) denote the neighbor off the triangle.
  - If \( d(w) \geq 6 \), then \( u \) gets all needed charge from \( uw \) and \( w \).
  - If \( d(v_1) \geq 12 \), then \( u \) gets all needed charge from \( uw \) and \( v_1 \).
  - (Similarly, if \( d(v_1) + d(v_2) \geq 16 \), then \( u \) gets enough charge).
- “Interesting” cases are when \( d(w) \) and \( d(v_1) + d(v_2) \) are small.
2-vertices

- A 2-vertex $u$ has initial charge $2 - 4 = -2$. 

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2-vertices

- A 2-vertex $u$ has initial charge $2 - 4 = -2$.
- First, if $u$ is on a triangle, then the degree sums of its two neighbors must be at least $\Delta + 5$.
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- A 2-vertex $u$ has initial charge $2 - 4 = -2$.
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![Diagram of a 2-vertex $u$ on a triangle with degree sums of its neighbors at least $\Delta + 5$.]
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All needed charge comes via R2 and R3.
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All needed charge comes via R2 and R3. We need $2 \left( \frac{\Delta - 4}{\Delta} \right) + \frac{1}{4} \geq 2$, which is true when $\Delta \geq 32$. 
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Otherwise, $u$ is not on a triangle, so it gets charge through its edges via R1.
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![Diagram showing a triangle and an edge](image)

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- Otherwise, $u$ is not on a triangle, so it gets charge through its edges via R1. We show that $u$ gets charge 1 from each side.
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- So all 3-vertices and 2-vertices end with nonnegative final charge, and everything works!
Summary

- Coloring the square of a planar graph $G$ requires a number of colors linear in $\Delta$, the maximum degree of $G$. 
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- For planar graphs without 4-cycles or 5-cycles, an upper bound of $\Delta + 5$ has been proven whenever $\Delta \geq 9$. 

We use discharging: assuming a minimal counterexample $G$, the initial charges sum to $-8$. After moving charge around, the final charges are all nonnegative, a contradiction! Hence no counterexample can exist.
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THANK YOU