Incidence Numbers and Hard Erdős Problems in Discrete Geometry

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About 20 years ago Laci Székely submitted a paper titled

*Crossing Numbers and Hard Erdős Problems in Discrete Geometry*

The method Laci introduced in that seminal paper had a huge impact on discrete geometry and it became a standard tool.

In this talk I will mention problems which were asked by Laci or which are related to his work.
Székely’s proof of the Szemerédi-Trotter theorem used the bound on crossing numbers.

**Theorem (Szemerédi-Trotter)**

The number of incidences between $n$ points and $m$ lines in $\mathbb{R}^2$ is $O(n^{2/3}m^{2/3} + n + m)$.

It was expected that similar bound holds over $\mathbb{C}^2$. But $\mathbb{C}^2$ has a very different geometry than $\mathbb{R}^2$. 
For complex points and lines a Szemerédi-Trotter type bound was proved by Tóth and later by Zahl. For "pseudolines" even in higher complex dimensions a similar bound holds (S.-Tao). Still, it would be interesting to find a Crossing Lemma type bound.

Theorem (Ajtai, Chvátal, Newborn, and Szemerédi)

For an undirected simple graph $G$ with $n$ vertices and $e$ edges such that $e > 7n$ then

$$cr(G) \geq \frac{e^3}{29n^2}$$
In $\mathbb{R}^2$ the key is that if more than $3n - 6$ point-pairs are connected by line segments then two will intersect.

What is an interval in $\mathbb{C}^2$? There are ways to define it, here is one which would be useful. Any two points $a$ and $b$ define a unique line. It is a plane (2-dimensional affine subspace) in $\mathbb{R}^4 \sim \mathbb{C}^2$. One can consider the disk on this plane with $a$ and $b$ being the antipodal points on the boundary of the disk.

$$l(a, b) = \left\{ \frac{a + b}{2} + z \left| z \right| \leq \left| \frac{a - b}{2} \right| \right\}$$
Crossings in $\mathbb{C}^2$ (cont’d)

$l(a, b) = \left\{ \frac{a + b}{2} + z \mid |z| \leq \left| \frac{a - b}{2} \right| \right\}$

**Conjecture**

There is a constant $C \in \mathbb{R}$ such that if more than $Cn$ point-pairs are connected by line segments in $\mathbb{C}^2$ then two of the segments will intersect.
Crossings in $\mathbb{C}^2$ (cont’d)

**Conjecture**

If $K_8$ is embedded by line segments into $\mathbb{C}^2$ then two of the segments will intersect.

Here is a $K_7$ embedded into $\mathbb{C}^2$ without crossing. (S. Leppänen)

\[
\begin{align*}
z_1 &= (0.4358 - 0.3796i, 0.5726 + 0.3896i) \\
z_2 &= (-0.3382 + 0.0719i, -0.1316 + 0.3220i) \\
z_3 &= (0.6391 + 0.0141i, 0.8889 - 0.3292i) \\
z_4 &= (0.6302 - 0.5513i, 0.2813 - 0.8285i) \\
z_5 &= (0.9731 - 1.3291i, 2.3615 + 0.4571i) \\
z_6 &= (1.7105 - 0.7780i, -1.4009 - 0.8982i) \\
z_7 &= (0.0099 - 0.9417i, 1.3350 - 0.9040i)
\end{align*}
\]
Problem of de Caen and Székely: How many triangles are there in a point-line arrangement?

**Theorem (Szemerédi-Trotter)**

*The number of incidences between n points and m lines in* $\mathbb{R}^2$ *is* $O\left(\frac{n^2}{3} \cdot \frac{m^2}{3} + n + m\right)$.

There are SOME triangles

**Theorem (S.)**

*For any* $c > 0$ *there is an* $n_0$ *such that if* $n > n_0$ *and the number of incidences between* $n$ *points and* $n$ *lines is at least* $cn^{4/3}$ *then the arrangement contains triangles.*
Questions.

- Is there a simple proof for the existence of triangles? (I used Szemerédi’s Regularity Lemma)

- What is the number of triangles under the conditions of the previous theorem? I can show $\Omega(n^{5/3})$ but it might be $\Omega(n^2)$.

- Maybe the following is true: If $m$ lines and $n$ points determine $cn^{2/3}m^{2/3}$ incidences then the arrangement contains $\Omega(nm)$ triangles. (Reverse de Caen-Székely)
Triple grids, another problem of Székely Solymosi

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Incidences
Using some heavy tools Elekes, Simonovits, and Szabó proved a subquadratic upper bound.

**Theorem (E-S-Sz)**

The number of triple points is $O(n^{1.99})$

A simplified method gave a better bound

**Theorem (Raz, Sharir, S.)**

The number of triple points is $O(n^{11/6})$

Maybe the truth is $O(n^{1+\epsilon})$. 
There is a more general algebraic framework behind the solutions. If a low degree algebraic surface has a large intersection with a Cartesian product then it has a special additive or multiplicative structure. Improving earlier result of Elekes and Rónyai we proved the following.
Algebraic Surface

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Incidences
Theorem (Raz, Sharir, S.)

For every constant-degree bivariate real polynomial \( f \), either
\[
|f(A, B)| = O(n^{4/3}), \quad \text{for every pair of finite sets } A, B \subset \mathbb{R}, \text{ with } |A| = |B| = n, \quad \text{or else } f \text{ must be of one of the special forms}
\]
\[
f(u, v) = h(\phi((u) + \rho(v)), \quad \text{or } f(u, v) = h(\phi(u)\rho(v)), \quad \text{for some univariate polynomials } \phi, \rho, h \text{ over } \mathbb{R}.
\]

This general bound and its extensions have many applications. It is not known whether \( O(n^{4/3}) \) could be replaced with \( O(n^{1+\varepsilon}) \) in the theorem above.
More algebra ...

**Definition**

Given an arrangement of lines in $\mathbb{R}^3$. A point which is a crossing point of three non co-planar lines is called a *joint*.

**Theorem (Guth and Katz)**

*The number of joints determined by $n^2$ lines is $O(n^3)$.*

This was the first in a sequence of results which define modern Discrete Geometry. Let us see the proof.
Claim: There is an algebraic surface \( f \), of degree \( d \leq 3n \) which contains all \( n^2 \) lines.

\[
f(x, y, z) = \sum_{0 \leq i+j+k \leq d} a_{ijk} x^i y^j z^k
\]

\[
\ell = \{(t, at + b, ct + d) | t \in \mathbb{R}\}
\]

The surface contains the line if after the substitution 
\( x = t, y = at + b, z = ct + d \) the polynomial (in \( t \)) is everywhere zero, so the coefficients are zero. We have \( n^2 d \) linear equations for the \( \binom{d+3}{3} \) variables. If

\[
n^2 d < \binom{d+3}{3}
\]

then there is a solution.
We will show that there is a line with a few (< 3n ) joints only. Let us choose the minimum degree surface which contains all lines. Note that \( \nabla f(x, y, z) = 0 \) whenever \( x, y, z \) is a joint. By Bezout’s theorem every line with at least \( d \leq 3n \) joints on it is contained by the degree \( d - 1 \) surface

\[
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.
\]

If all lines had \( 3n \) or more joints then all lines would lie in a smaller degree surface which would be a contradiction.
Theorem

Lines in $\mathbb{R}^3$ If $n^2$ lines have more than $n^3$ crossings then at least $n$ of them are contained by a doubly ruled surface.
Doubly Ruled

A doubly ruled elliptic hyperboloid.

(1) Cylinder
(2) Draw off the lines slightly shifted
(3) A hyperboloid is formed

Fig. 1 Hyperboloid
A doubly ruled hyperbolic paraboloid.
Theorem (Guth-Zahl, bipartite version)

Let $\mathcal{C}, \mathcal{C}'$ be constructible sets of curves. Let $\mathcal{L} \subset \mathcal{C}$, $\mathcal{L}' \subset \mathcal{C}'$ be finite sets of curves, each of cardinality at most $n$. Then at least one of the following must hold:

- The number of two–rich points is $O(n^{3/2})$, where the implicit constant depends only on $\mathcal{C}$ and $\mathcal{C}'$.
- There is an irreducible surface $Z \subset \mathbb{C}^3$ that is doubly ruled by curves from $\mathcal{C}$ and $\mathcal{C}'$. 
In this talk we sketched some ideas and proofs. For more details check the papers below.