1. The geometry of Cartesian Products - Solymosi talk 2

**Theorem 1.1** For all $m \in \mathbb{N}$ there is a $\delta_m$ such that if $|A + A| \leq |A|^{1+\delta_m}$ implies $A$ contains a $m$ Hilbert cube.

*Proof.* $A \times A$ can be covered by $\leq |A|^{1+\delta_m}$ lines. Then we may repeat what we did in the first lecture for the proof of the corner theorem. For $m = 2$, looking for a binary tree of depth 2 to find 3 Hilbert cubes (actually many of them). □

Now we move onto hyper elliptic curves. Consider a polynomial of the form

$$y^2 = f(x),$$

where the degree of $f \in \mathbb{Z}[x]$ of degree at least 5 with distinct roots. If the degree is $2g + 1$ or $2g + 2$, then the genus is $g$. We have the following famous conjecture from number theory.

**Conjecture 1.2** *(Bombieri–Lang conjecture)* The number of rational points on a hyper elliptic is uniformly bounded by the genus.

It is highly recommended by Jozsef to look at Walter Rudin’s paper, ”Trigonometric series with gaps.” One conjecture is that for a finite subset of the squares in the integers,

$$|A + A| \geq |A|^{2-\epsilon}. \quad (1)$$

It turns out that the Bombieri–Lang conjecture can provide information about the problem in (1). Note if a set has small doubling, one can find many Hilbert cubes, as above. Suppose $X, X + x_1, X + x_2, X + x_1 + x_2 \in A$ which is a subset of the squares. Then their product is a square. Write $X = x^2$.

Then

$$(x^2 + x_1)(x^2 + x_2)(x^2 + x_1 + x_2) = y^2,$$

which only has a bounded number of solutions, thanks to the Bombieri–Lang conjecture. Thus $A$ can only have a few Hilbert cubes, which contradicts the (proof of) Theorem 1.1 which says a set with small doubling has many Hilbert cubes. This shows that for any subset of the squares

$$|A + A| \geq |A|^{1+\delta}.$$
class is the set of lines. Join \((a, b)\) to a line if they are incident. Then the number of incidences is at most the number of edges in a \(C_4\) free graph, which is known to be
\[
\min\{\|L\|^{1/2}\|A\|\|B\|, (\|A\|\|B\|)^{1/2}\|L\|\}.
\]
This cannot be improved in general, but there are a host of results in various fields. \(\text{Szemerédi–Trotter}\) over \(\mathbb{R}\) and this paper of Stevens and Zeeuw over \(\mathbb{F}_p\).

We now look at a proof of Szemerédi–Trotter for cartesian products. It is useful to introduce \(k\)-rich points. For a point \((a, b)\) we say it is \(k\)-rich if there are at least \(k\) lines passing through \((a, b)\).

**Theorem 1.3 (Szemerédi–Trotter, special case)** The number of \(k\) rich lines in \(A \times B \subset \mathbb{R}^2\) is at most
\[
C|A|^2|B|^{1/2}k^{-3}.
\]
Before proving this theorem, let’s see a beautiful application to number theory. The following theorem was contained in a 2.5 page paper of Elekes.

**Theorem 1.4 (Elekes)** Let \(A \subset \mathbb{R}\) be finite. Then
\[
|A + A| + |AA| \gg |A|^{5/4}.
\]
**Proof.** Consider \(A \times A\) and consider the \(|A|^2\) lines \(y = a(x - c)\). Then every line contains the \(|A|\) points \((b + c, ab)\). Thus every line is \(|A|\)-rich. Applying Szemerédi–Trotter, we find
\[
|A|^2 \leq |A + A|^2|AA|^2|A|^{-3}.
\]

Now onto the proof of Szemer’edi–Trotter.

**Proof.** This proof can be adjusted to unequal parts, but let’s consider equal parts for the sake of simplicity.

Cut the sets \(A\) and \(B\) into equal consecutive parts of size \(k/4\). This partitions \(A \times B\) into \(k^2/16\) rectangles (cells). Any \(k\)-rich line will intersect at most \(k/4\) horizontal and \(k/4\) vertical lines. The number of “lucky” pairs of lines (that is two consecutive points of a line that lie in the same rectangle) is at least \(|X_k|k/2\), where \(X_k\) is the number of \(k\)-rich lines. But the number of lucky pairs is at most \(16|A||B|k^{-2}\). Simplification gives the desired bound. \(\Box\)

For a probabilistic interpretation of why one would want to break the grid up, see this post of Tao.

Note this proof works for curves of bounded degree. There is a nice open question (on the “homework”), what is the maximal number of 4-tuples which are on the same circle. On the other hand, it easily follows that the number of collinear triples is at most \(O(|A|^2|B|^2\log(|A| + |B|))\).