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COMPACTLY SUPPORTED FRAMES FOR SPACES OF DISTRIBUTIONS IN THE FRAMEWORK OF DIRICHLET SPACES

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Abstract. A small perturbation method is developed and deployed to the construction of compactly supported frames for Besov and Triebel-Lizorkin in the general setting of Dirichlet space with a doubling measure and local scale-invariant Poincaré inequality. This allows, in particular, to develop compactly supported frames for Besov and Triebel-Lizorkin spaces in the context of Lie groups, Riemannian manifolds, and various other settings. The compactly supported frames are utilized for the development of atomic Hardy spaces H^p_A in the general framework of Dirichlet spaces.

1. Introduction

Compactly supported frames and bases are an important tool in Harmonic analysis and its applications in allowing to represent functions and distributions in terms of building blocks of small supports. The atomic decompositions exhibit another side of the same idea. The purpose of this study is to construct frames with compactly supported frame elements of small shrinking supports in the general framework of Dirichlet spaces, described in [2, 12]. More explicitly, compactly supported frames will be developed in the general setting of strictly local regular Dirichlet spaces with doubling measure and local scale-invariant Poincaré inequality, leading to a Markovian heat kernel with small time Gaussian bounds and Hölder continuity. In particular, this theory allows to develop compactly supported frames on Lie groups or homogeneous spaces with polynomial volume growth, complete Riemannian manifolds with Ricci curvature bounded from below and satisfying the volume doubling condition, and in various other nonclassical setups. Naturally, it covers the more classical cases on the sphere, interval, ball, and simplex with weights.

Compactly supported frames have already been constructed on the sphere in [16] and on the ball with weight $w_\mu(x) = (1 - |x|)^{\mu-1/2}$, where μ is a half integer and $\mu \geq 0$ in [17]. One of the strengths of our method is that although it is general it allows to obtain in particular settings better results than the existing ones. For example, combining results from this article and [13] enable us to improve the results on the ball from [17] by relaxing the condition on μ from a half integer and $\mu \geq 0$ to any $\mu > -1/2$. Another application of the results from the current

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paper and [13] is to the development of compactly supported frames on the interval with Jacobi weights and on the simplex with weights.

A key feature of the new frames is that they can be used for decomposition of the Besov and Triebel-Lizorkin spaces in the general framework of Dirichlet spaces developed in [12], and therefore, in many particular settings of interest.

An important application of the compactly supported frames from this article is to atomic Hardy spaces H^p_A, $0 < p \le 1$. The compactly supported frames provide a vehicle in establishing Littlewood-Play characterization of the Hardy spaces H^p_A and their frame decomposition.

We shall operate in the setting established in [2, 12], which we next recall briefly:

I. We assume that (M, ρ, μ) is a metric measure space satisfying the conditions: (M, ρ) is a locally compact metric space with distance $\rho(\cdot, \cdot)$ and μ is a positive Radon measure such that the following *volume doubling condition* is valid

$$0 < \mu(B(x, 2r)) \le c_0 \mu(B(x, r)) < \infty \quad \text{for all } x \in M \text{ and } r > 0, \tag{1.1}$$

where $B(x, r)$ is the open ball centered at x of radius r and $c_0 > 1$ is a constant. The above yields

$$\mu(B(x, \lambda r)) \le c_0 \lambda^d \mu(B(x, r)) \quad \text{for } x \in M,\ r > 0, \text{ and } \lambda > 1, \tag{1.2}$$

were $d = \log_2 c_0 > 0$ is a constant playing the role of a dimension.

II. The main assumption is that the local geometry of the space (M, ρ, μ) is related to an essentially self-adjoint positive operator L on $L^2(M, d\mu)$ such that the associated semigroup $P_t = e^{-tL}$ consists of integral operators with (heat) kernel $p_t(x, y)$ obeying the conditions:

• *Small time Gaussian upper bound*:

$$|p_t(x, y)| \le \frac{C^\star \exp\{-\frac{c^\star \rho^2(x,y)}{t}\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \quad \text{for } x, y \in M,\ 0 < t \le 1. \tag{1.3}$$

• *Hölder continuity*: There exists a constant $\alpha > 0$ such that

$$\big|p_t(x, y) - p_t(x, y')\big| \le C^\star \Big(\frac{\rho(y, y')}{\sqrt{t}}\Big)^\alpha \frac{\exp\{-\frac{c^\star \rho^2(x,y)}{t}\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \tag{1.4}$$

for $x, y, y' \in M$ and $0 < t \le 1$, whenever $\rho(y, y') \le \sqrt{t}$.

• *Markov property*:

$$\int_M p_t(x, y) d\mu(y) \equiv 1 \quad \text{for } t > 0. \tag{1.5}$$

Above $C^\star, c^\star > 0$ are structural constants.

We shall also assume the following *additional conditions*:

• *Non-collapsing condition*: There exists a constant $c > 0$ such that

$$\inf_{x \in M} \mu(B(x, 1)) \ge c. \tag{1.6}$$

• *Reverse doubling condition*: There exists a constant $c > 1$ such that

$$\mu(B(x, 2r)) \ge c\mu(B(x, r)) \quad \text{for } x \in M \text{ and } 0 < r \le \tfrac{\operatorname{diam} M}{3}. \tag{1.7}$$

The latter condition is only needed for lower bound estimates on the L^p-norms of the frame elements (see Proposition 2.5). It can be relaxed if such estimates are not needed, which is the case in the general theory.

A natural effective realization of the above setting appears in the general framework of Dirichlet spaces. More precisely, in the framework of strictly local regular Dirichlet spaces with a complete intrinsic metric it suffices to only verify the local Poincaré inequality and the global doubling condition on the measure and then the above general setting applies in full. For more details, see [2]. The key observation is that situations where our theory applies are quite common, which becomes evident from the examples given in [2].

We next outline the main points in this paper. We build on results on functional calculus, frames and spaces of distributions developed in [2, 12]. For convenience, in §2 we collect all the results we need from [2, 12].

To achieve our goals we first develop in §3 a general small perturbation scheme for construction of frames in a general quasi-Banach space $\mathcal{B}$ of distributions given a pair of dual frames $\{\psi_\xi\}$, $\{\tilde{\psi}_\xi\}$. In fact, this is the situation in [12]. Such a method has been developed in [16] in the more favorable situation when a single frame $\{\psi_\xi\}$ for $\mathcal{B}$ exists (see §3.3). The latter scheme can be applied directly in our setting in the spacial case when the spectral spaces have the polynomial property (see [12]) as on the sphere, interval, ball, and simplex. The idea of these schemes is rooted in the development of bases in [22], also in [14, 15], and is related to the method for construction of atomic decompositions in [1].

The construction of compactly supported frames in the current setting is given in §4. It relies heavily on the *finite speed propagation property* of solutions of the wave equation associated with the operator L, see (2.5) below. This property follows from the Gaussian bound (1.3) on the heat kernel $p_t(x, y)$. The finite speed propagation property alone, however, is not sufficient. The other properties of the heat kernel and the doubling condition on the measure given above are also important for the development of a complete theory. In particular, they allowed to develop in [12] Besov and Triebel-Lizorkin spaces with full set of indices and their frame characterization, which play a critical role here.

In §5 the compactly supported frames from §4 are applied to the development of the atomic Hardy spaces H^p_A in the setting of this article.

In §6 the developments from previous sections are applied to specific settings on the interval, ball, and simplex.

Section 7 is an appendix where we place the proof of the boundedness of almost diagonal operators on Besov and Triebel-Lizorkin sequence spaces.

Some useful notation: Throughout we shall denote $|E| := \mu(E)$ and $\mathbb{1}_E$ will stand for the characteristic function of $E \subset M$, $\|\cdot\|_p = \|\cdot\|_{L^p} := \|\cdot\|_{L^p(M,d\mu)}$. Positive constants will be denoted by c, C, c_1, c', $\dots$ and will be allowed to vary at every occurrence. The notation $a \sim b$ will stand for $c_1 \le a/b \le c_2$. We shall also use the standard notation $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

2. Background

In developing compactly supported frames we shall make extensive use of results from [2, 12]. In this section we review everything that will be needed from [2, 12].

2.1. **Functional calculus.** We adhere to the notation in [2, 12]. In particular, the following symmetric functions will appear in the following:

$$D_{\delta,\sigma}(x, y) := \big(|B(x, \delta)||B(y, \delta)|\big)^{-1/2}\Big(1 + \frac{\rho(x, y)}{\delta}\Big)^{-\sigma}, \quad x, y \in M. \tag{2.1}$$

As $B(x,r)\subset B(y,\rho(y,x)+r)$, (1.2) yields

$$|B(x,r)|\le c_0\Big(1+\frac{\rho(x,y)}{r}\Big)^d|B(y,r)|,\quad x,y\in M,\ r>0. \tag{2.2}$$

Combining this with (2.1) we arrive at this useful inequality

$$D_{\delta,\sigma}(x,y)\le c_0^{1/2}|B(x,\delta)|^{-1}\Big(1+\frac{\rho(x,y)}{\delta}\Big)^{-\sigma+d/2}. \tag{2.3}$$

Here $|B(x,\delta)|^{-1}$ on the right can be replaced by $|B(y,\delta)|^{-1}$.

The following inequality will be instrumental in some proofs [12, Lemma 2.1]: For $\sigma>d$ and $\delta>0$

$$\int_M\big(1+\delta^{-1}\rho(x,y)\big)^\sigma d\mu(y)\le c|B(x,\delta)|,\quad x\in M. \tag{2.4}$$

The **finite speed propagation property** will play a key role in this study:

$$\big\langle\cos(t\sqrt{L})f_1,f_2\big\rangle=0,\quad 0<\tilde{c}t<r,\quad \tilde{c}:=\frac{1}{2\sqrt{c^\star}}, \tag{2.5}$$

for all open sets $U_j\subset M$, $f_j\in L^2(M)$, $\operatorname{supp}f_j\subset U_j$, $j=1,2$, where $r:=\rho(U_1,U_2)$.

This property implies the following localization result for the kernels of operators of the form $f(\delta\sqrt{L})$ whenever $\hat{f}$ is band limited. Here $\hat{f}(\xi):=\int_{\mathbb{R}}f(t)e^{-it\xi}dt$.

Proposition 2.1. *Let f be even,* $\operatorname{supp}\hat{f}\subset[-A,A]$ *for some $A>0$, and $\hat{f}\in W^2_\infty$, i.e. $\|\hat{f}^{(2)}\|_\infty<\infty$. Then for $\delta>0$ and $x,y\in M$*

$$f(\delta\sqrt{L})(x,y)=0\quad \textit{if}\quad \rho(x,y)>\tilde{c}\delta A. \tag{2.6}$$

We shall need the following result from the smooth functional calculus induced by the heat kernel developed in [12] (Theorem 3.1).

Theorem 2.2. [12] *Let $f\in C^k(\mathbb{R}_+)$, $k\ge d+1$,* $\operatorname{supp}f\subset[0,R]$ *for some $R\ge1$, and $f^{(2\nu+1)}(0)=0$ for $\nu\ge0$ such that $2\nu+1\le k$. Then $f(\delta\sqrt{L})$, $0<\delta\le1$, is an integral operator with kernel $f(\delta\sqrt{L})(x,y)$ satisfying*

$$\big|f(\delta\sqrt{L})(x,y)\big|\le c_kD_{\delta,k}(x,y)\quad \textit{and} \tag{2.7}$$

$$\big|f(\delta\sqrt{L})(x,y)-f(\delta\sqrt{L})(x,y')\big|\le c_k'\Big(\frac{\rho(y,y')}{\delta}\Big)^\alpha D_{\delta,k}(x,y)\ \ \textit{if}\ \ \rho(y,y')\le\delta. \tag{2.8}$$

Here $D_{\delta,k}(x,y)$ is from (2.1),

$$c_k=c_k(f)=R^d\big[(c_1k)^k\|f\|_{L^\infty}+(c_2R)^k\|f^{(k)}\|_{L^\infty}\big],\quad c_k'=c_3c_kR^\alpha, \tag{2.9}$$

where $c_1,c_2,c_3>0$ depend only on the constants $c_0,C^\star,c^\star$ from (1.1) − (1.4)*, c_3 depends on k as well; $\alpha>0$ is the constant from* (1.4)*. Furthermore,*

$$\int_M f(\delta\sqrt{L})(x,y)d\mu(y)=f(0). \tag{2.10}$$

This theorem readily implies the following result that will be needed later on.

Corollary 2.3. *Let $f\in C^\infty(\mathbb{R}_+)$,* $\operatorname{supp}f\subset[0,R]$*, $R\ge1$, and $f^{(2\nu+1)}(0)=0$, $\nu\ge0$. Then for any $n\ge0$ and $0<\delta\le1$ the operator $L^nf(\delta\sqrt{L})$ is an integral operator with kernel $L^nf(\delta\sqrt{L})(x,y)$ having the property that for any $\sigma>0$ there exists a constant $c_{\sigma,n}>0$ such that*

$$\big|L^nf(\delta\sqrt{L})(x,y)\big|\le c_{\sigma,n}\delta^{-2n}D_{\delta,\sigma}(x,y),\quad x,y\in M. \tag{2.11}$$

The requirement in Theorem 2.2 that f is compactly supported can be relaxed.

Theorem 2.4. [12] *Suppose* $f \in C^k(\mathbb{R}_+)$, $k \ge d+1$,

$$|f^{(\nu)}(\lambda)| \le C_k(1+\lambda)^{-r} \text{ for } \lambda > 0 \text{ and } 0 \le \nu \le k, \text{ where } r \ge k+d+1,$$

and $f^{(2\nu+1)}(0) = 0$ *for* $\nu \ge 0$ *such that* $2\nu+1 \le k$. *Then* $f(\delta\sqrt{L})$ *is an integral operator with kernel* $f(\delta\sqrt{L})(x,y)$ *satisfying* (2.7)-(2.8), *where the constants* c_k, c_k' *depend on* k, d, α, $c_0, C^\star, c^\star$, *and linearly on* C_k.

2.2. **Spectral spaces.** Let E_λ, $\lambda \ge 0$, be the spectral resolution associated with the self-adjoint positive operator L on $L^2 := L^2(M, d\mu)$. We let F_λ, $\lambda \ge 0$, denote the spectral resolution associated with $\sqrt{L}$, i.e. $F_\lambda = E_{\lambda^2}$. Then for any measurable and bounded function f on $\mathbb{R}_+$ the operator $f(\sqrt{L})$ is defined by $f(\sqrt{L}) = \int_0^\infty f(\lambda) dF_\lambda$ on L^2. For the spectral projectors we have $E_\lambda = \mathbb{1}_{[0,\lambda]}(L) := \int_0^\infty \mathbb{1}_{[0,\lambda]}(u) dE_u$ and

$$F_\lambda = \mathbb{1}_{[0,\lambda]}(\sqrt{L}) := \int_0^\infty \mathbb{1}_{[0,\lambda]}(u) dF_u = \int_0^\infty \mathbb{1}_{[0,\lambda]}(\sqrt{u}) dE_u. \tag{2.12}$$

For any compact $K \subset [0,\infty)$ the spectral space Σ_K^p is defined by

$$\Sigma_K^p := \{f \in L^p : \theta(\sqrt{L})f = f \text{ for all } \theta \in C_0^\infty(\mathbb{R}_+),\ \theta \equiv 1 \text{ on } \ K\}.$$

In general, given a space Y of measurable functions on M we set

$$\Sigma_\lambda = \Sigma_\lambda(Y) := \{f \in Y : \theta(\sqrt{L})f = f \text{ for all } \theta \in C_0^\infty(\mathbb{R}_+),\ \theta \equiv 1 \text{ on } \ [0,\lambda]\}.$$

2.3. **Frames.** Our construction of compactly supported frames will rely on the frames developed in [12]. Here we collect the needed information from [12].

Construction of Frame # 1. The construction begins with a cut-off function Φ with the following properties: $\Phi \in C^\infty(\mathbb{R}_+)$, $\Phi(u) = 1$ for $u \in [0,1]$, $0 \le \Phi \le 1$, and $\operatorname{supp}\Phi \subset [0,b]$, where $b > 1$ is a constant, see [12]. We shall assume that $b \ge 2$. Set $\Psi(u) := \Phi(u) - \Phi(bu)$.

An important point is that the function Φ can be selected so that the operators $\Phi(\delta\sqrt{L})$ and $\Psi(\delta\sqrt{L})$ are integral operators whose kernels $\Phi(\delta\sqrt{L})(x,y)$ and $\Psi(\delta\sqrt{L})(x,y)$ have sub-exponential space localization, namely,

$$|\Phi(\delta\sqrt{L})(x,y)|, |\Psi(\delta\sqrt{L})(x,y)| \le c\frac{\exp\big\{-\kappa\big(\frac{\rho(x,y)}{\delta}\big)^\beta\big\}}{\big(|B(x,\delta)||B(y,\delta)|\big)^{1/2}}, \quad x,y \in M, \tag{2.13}$$

where $0 < \beta < 1$, $\kappa, c > 0$, and β can be selected as close to 1 as we wish. Furthermore, $\Phi(\delta\sqrt{L})(x,y)$ and $\Psi(\delta\sqrt{L})(x,y)$ are Hölder continuous (see [12]).

Setting

$$\Psi_0(u) := \Phi(u) \quad \text{and} \quad \Psi_j(u) := \Psi(b^{-j}u),\ \ j \ge 1, \tag{2.14}$$

we have $\Psi_j \in C^\infty(\mathbb{R}_+)$, $0 \le \Psi_j \le 1$, $\operatorname{supp}\Psi_0 \subset [0,b]$, $\operatorname{supp}\Psi_j \subset [b^{j-1}, b^{j+1}]$, $j \ge 1$, and $\sum_{j\ge0}\Psi_j(u) = 1$ for $u \in \mathbb{R}_+$. Hence we have the following Littlewood-Paley decomposition

$$f = \sum_{j\ge0}\Psi_j(\sqrt{L})f \quad \text{for } f \in \mathcal{D}' \text{ (and } f \in L^p). \tag{2.15}$$

For $j \ge 0$ we let $\mathcal{X}_j \subset M$ be a maximal δ_j−net on M with $\delta_j := \gamma b^{-j-2}$ and suppose $\{A_\xi\}_{\xi\in\mathcal{X}_j}$ is a companion disjoint partition of M consisting of measurable sets such that $B(\xi,\delta_j/2)\subset A_\xi\subset B(\xi,\delta_j)$, $\xi\in\mathcal{X}_j$. Here $\gamma>0$ is a sufficiently small constant.

The jth level frame elements ψ_ξ are defined by

$$\psi_\xi(x):=|A_\xi|^{1/2}\Psi_j(\sqrt{L})(x,\xi),\quad \xi\in\mathcal{X}_j. \tag{2.16}$$

Let $\mathcal{X}:=\cup_{j\ge0}\mathcal{X}_j$, where equal points from different sets $\mathcal{X}_j$ will be regarded as distinct elements of $\mathcal{X}$, so $\mathcal{X}$ can be used as an index set. Then $\{\psi_\xi\}_{\xi\in\mathcal{X}}$ is Frame #1.

The construction of a dual frame $\{\tilde\psi_\xi\}_{\xi\in\mathcal{X}}$ is much more involved; we refer the reader to §4.3 in [12] for the details.

We next describe the main properties of $\{\psi_\xi\}_{\xi\in\mathcal{X}}$ and $\{\tilde\psi_\xi\}_{\xi\in\mathcal{X}}$.

Proposition 2.5. [12] (a) Localization: *For any $0<\hat\kappa<\kappa/2$ there exist a constant $\hat c>0$ such that for any $\xi\in\mathcal{X}_j$, $j\ge0$,*

$$|\psi_\xi(x)|,|\tilde\psi_\xi(x)|\le\hat c|B(\xi,b^{-j})|^{-1/2}\exp\big\{-\hat\kappa(b^j\rho(x,\xi))^\beta\big\} \tag{2.17}$$

and for any $m\ge1$

$$|L^m\psi_\xi(x)|,|L^m\tilde\psi_\xi(x)|\le c_m|B(\xi,b^{-j})|^{-1/2}b^{2jm}\exp\big\{-\hat\kappa(b^j\rho(x,\xi))^\beta\big\}. \tag{2.18}$$

Also, if $\rho(x,y)\le b^{-j}$, then

$$|\psi_\xi(x)-\psi_\xi(y)|\le\hat c|B(\xi,b^{-j})|^{-1/2}(b^j\rho(x,y))^\alpha\exp\big\{-\hat\kappa(b^j\rho(x,\xi))^\beta\big\} \tag{2.19}$$

and the same inequality holds for $\tilde\psi_\xi$.

(b) Norms: *If in addition the reverse doubling condition* (1.7) *is valid, then*

$$\|\psi_\xi\|_p\sim\|\tilde\psi_\xi\|_p\sim|B(\xi,b^{-j})|^{\frac1p-\frac12},\quad 0<p\le\infty. \tag{2.20}$$

(c) Spectral localization: $\psi_\xi,\tilde\psi_\xi\in\Sigma^p_b$ *if* $\xi\in\mathcal{X}_0$, $\psi_\xi\in\Sigma^p_{[b^{j-1},b^{j+1}]}$ *if* $\xi\in\mathcal{X}_j$, *and* $\tilde\psi_\xi\in\Sigma^p_{[b^{j-2},b^{j+2}]}$ *if* $\xi\in\mathcal{X}_j$, $j\ge1$, $0<p\le\infty$.

(d) Representation: *For any $f\in\mathcal{D}'$ we have*

$$f=\sum_{\xi\in\mathcal{X}}\langle f,\tilde\psi_\xi\rangle\psi_\xi=\sum_{\xi\in\mathcal{X}}\langle f,\psi_\xi\rangle\tilde\psi_\xi\quad in\ \mathcal{D}'. \tag{2.21}$$

This also holds for $f\in L^p$, $1\le p\le\infty$, with the usual modification when $p=\infty$.

(e) *Each of the systems $\{\psi_\xi\}$ and $\{\tilde\psi_\xi\}$ is a frame for L^2.*

2.4. **Besov and Triebel-Lizorkin spaces.** The Besov and Triebel-Lizorkin spaces associated with the operator L, defined in [12], are in general spaces of distributions. There are some distinctions, however, between the tests functions and distributions that are used depending on whether $\mu(M)<\infty$ or $\mu(M)=\infty$.

In the case $\mu(M)<\infty$, we use as test functions the class $\mathcal{D}$ of all functions $\phi\in\cap_{m\ge0}D(L^m)$ with the topology induced by

$$\mathcal{P}_m(\phi):=\|L^m\phi\|_2,\quad m\ge0. \tag{2.22}$$

If $\mu(M)=\infty$, then the class of test functions $\mathcal{D}$ is defined as the set of all functions $\phi\in\cap_{m\ge0}D(L^m)$ such that

$$\mathcal{P}_{m,\ell}(\phi):=\sup_{x\in M}(1+\rho(x,x_0))^\ell|L^m\phi(x)|<\infty\quad\forall m,\ell\ge0. \tag{2.23}$$

Here $x_0\in M$ is selected arbitrarily and fixed once and for all.

As usual the space $\mathcal{D}'$ of distributions on M is defined as the set of all continuous linear functionals on $\mathcal{D}$ and the pairing of $f\in\mathcal{D}'$ and $\phi\in\mathcal{D}$ will be denoted by $\langle f,\phi\rangle:=f(\overline{\phi})$.

To handle possible anisotropic geometries there are two types of Besov (B) and Triebel-Lizorkin (F) spaces introduced in [12]: (i) classical B-spaces $B^s_{pq} = B^s_{pq}(L)$ and F-spaces $F^s_{pq} = F^s_{pq}(L)$, and (ii) nonclassical B-spaces $\tilde{B}^s_{pq} = \tilde{B}^s_{pq}(L)$ and F-spaces $\tilde{F}^s_{pq} = \tilde{F}^s_{pq}(L)$. We next recall them. Let the functions $\varphi_0, \varphi \in C^\infty(\mathbb{R}_+)$ be so that

$$\operatorname{supp}\varphi_0 \subset [0,2],\ \varphi_0^{(2\nu+1)}(0) = 0 \text{ for } \nu \ge 0,\ |\varphi_0(\lambda)| \ge c > 0 \text{ for } \lambda \in [0, 2^{3/4}], \tag{2.24}$$

$$\operatorname{supp}\varphi \subset [1/2,2],\ \ |\varphi(\lambda)| \ge c > 0 \text{ for } \lambda \in [2^{-3/4}, 2^{3/4}]. \tag{2.25}$$

Then $|\varphi_0(\lambda)| + \sum_{j\ge1} |\varphi(2^{-j}\lambda)| \ge c > 0$, $\lambda \in \mathbb{R}_+$. Set $\varphi_j(\lambda) := \varphi(2^{-j}\lambda)$ for $j \ge 1$.

Definition 2.6. *Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$.*

(i) The Besov space $B^s_{pq} = B^s_{pq}(L)$ is defined as the set of all $f \in \mathcal{D}'$ such that

$$\|f\|_{B^s_{pq}} := \Big(\sum_{j\ge0} \Big(2^{sj}\|\varphi_j(\sqrt{L})f(\cdot)\|_{L^p}\Big)^q\Big)^{1/q} < \infty. \tag{2.26}$$

(ii) The Besov space $\tilde{B}^s_{pq} = \tilde{B}^s_{pq}(L)$ is defined as the set of all $f \in \mathcal{D}'$ such that

$$\|f\|_{\tilde{B}^s_{pq}} := \Big(\sum_{j\ge0} \Big(\||B(\cdot, 2^{-j})|^{-s/d}\varphi_j(\sqrt{L})f(\cdot)\|_{L^p}\Big)^q\Big)^{1/q} < \infty. \tag{2.27}$$

Definition 2.7. *Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$.*

(a) The Triebel-Lizorkin space $F^s_{pq} = F^s_{pq}(L)$ is defined as the set of all $f \in \mathcal{D}'$ such that

$$\|f\|_{F^s_{pq}} := \Big\|\Big(\sum_{j\ge0} \Big(2^{js}|\varphi_j(\sqrt{L})f(\cdot)|\Big)^q\Big)^{1/q}\Big\|_{L^p} < \infty. \tag{2.28}$$

(b) The Triebel-Lizorkin space $\tilde{F}^s_{pq} = \tilde{F}^s_{pq}(L)$ is defined as the set of all $f \in \mathcal{D}'$ such that

$$\|f\|_{\tilde{F}^s_{pq}} := \Big\|\Big(\sum_{j\ge0} \Big(|B(\cdot, 2^{-j})|^{-s/d}|\varphi_j(\sqrt{L})f(\cdot)|\Big)^q\Big)^{1/q}\Big\|_{L^p} < \infty. \tag{2.29}$$

Above in both definitions the ℓ^q-norm is replaced by the sup-norm if $q = \infty$.

Frame decomposition of Besov and Triebel-Lizorkin spaces. One of the main result in [12] asserts that the Besov and Triebel-Lizorkin spaces from above can be characterized in terms of respective sequence norms of the frame coefficients of distributions, using the frames $\{\psi_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde{\psi}_\xi\}_{\xi\in\mathcal{X}}$ from §2.3.

To state this result we next introduce the sequence spaces b^s_{pq}, $\tilde{b}^s_{pq}$, and f^s_{pq}, $\tilde{f}^s_{pq}$, associated with the B- and F-spaces. As before $\mathcal{X} := \cup_{j\ge0}\mathcal{X}_j$ will denote the sets of the centers of the frame elements and $\{A_\xi\}_{\xi\in\mathcal{X}_j}$ will be the associated partitions of M.

Definition 2.8. *Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$.*

(a) b^s_{pq} is defined as the space of all complex-valued sequences $a := \{a_\xi\}_{\xi\in\mathcal{X}}$ such that

$$\|a\|_{b^s_{pq}} := \Big(\sum_{j\ge0} b^{jsq}\Big[\sum_{\xi\in\mathcal{X}_j} \Big(|B(\xi, b^{-j})|^{1/p-1/2}|a_\xi|\Big)^p\Big]^{q/p}\Big)^{1/q} < \infty. \tag{2.30}$$

(b) $\tilde{b}^s_{pq}$ *is defined as the space of all complex-valued sequences* $a := \{a_\xi\}_{\xi\in\mathcal{X}}$ *such that*

$$\|a\|_{\tilde{b}^s_{pq}} := \Big(\sum_{j\ge 0}\Big[\sum_{\xi\in\mathcal{X}_j}\Big(|B(\xi,b^{-j})|^{-s/d+1/p-1/2}|a_\xi|\Big)^p\Big]^{q/p}\Big)^{1/q} < \infty. \tag{2.31}$$

Definition 2.9. *Suppose* $s\in\mathbb{R}$, $0<p<\infty$, *and* $0<q\le\infty$.

(a) f^s_{pq} *is defined as the space of all complex-valued sequences* $a := \{a_\xi\}_{\xi\in\mathcal{X}}$ *such that*

$$\|a\|_{f^s_{pq}} := \Big\|\Big(\sum_{j\ge 0} b^{jsq}\sum_{\xi\in\mathcal{X}_j}\big[|a_\xi|\tilde{\mathbb{1}}_{A_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p} < \infty. \tag{2.32}$$

(b) $\tilde{f}^s_{pq}$ *is defined as the space of all complex-valued sequences* $a := \{a_\xi\}_{\xi\in\mathcal{X}}$ *such that*

$$\|a\|_{\tilde{f}^s_{pq}} := \Big\|\Big(\sum_{\xi\in\mathcal{X}}\big[|A_\xi|^{-s/d}|a_\xi|\tilde{\mathbb{1}}_{A_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p} < \infty. \tag{2.33}$$

Here $\tilde{\mathbb{1}}_{A_\xi} := |A_\xi|^{-1/2}\mathbb{1}_{A_\xi}$ *with* $\mathbb{1}_{A_\xi}$ *being the characteristic function of* A_ξ.

Above as usual the ℓ^p or ℓ^q norm is replaced by the sup-norm if $p=\infty$ or $q=\infty$.

In stating the results from [12] we shall use the "analysis" and "synthesis" operators defined by

$$S_{\tilde{\psi}} : f \to \{\langle f,\tilde{\psi}_\xi\rangle\}_{\xi\in\mathcal{X}} \quad\text{and}\quad T_\psi : \{a_\xi\}_{\xi\in\mathcal{X}} \to \sum_{\xi\in\mathcal{X}} a_\xi\psi_\xi. \tag{2.34}$$

Here the roles of $\{\psi_\xi\}$ and $\{\tilde{\psi}_\xi\}$ can be interchanged.

Theorem 2.10. [12] *Let* $s\in\mathbb{R}$ *and* $0<p,q\le\infty$. *(a) The operators* $S_{\tilde{\psi}} : B^s_{pq}\to b^s_{pq}$ *and* $T_\psi : b^s_{pq}\to B^s_{pq}$ *are bounded and* $T_\psi\circ S_{\tilde{\psi}} = Id$ *on* B^s_{pq}. *Consequently, for* $f\in\mathcal{D}'$ *we have* $f\in B^s_{pq}$ *if and only if* $\{\langle f,\tilde{\psi}_\xi\rangle\}_{\xi\in\mathcal{X}}\in b^s_{pq}$. *Moreover, if* $f\in B^s_{pq}$, *then* $\|f\|_{B^s_{pq}}\sim\|\{\langle f,\tilde{\psi}_\xi\rangle\}\|_{b^s_{pq}}$ *and under the reverse doubling condition* (1.7)

$$\|f\|_{B^s_{pq}} \sim \Big(\sum_{j\ge 0} b^{jsq}\Big[\sum_{\xi\in\mathcal{X}_j}\|\langle f,\tilde{\psi}_\xi\rangle\psi_\xi\|_p^p\Big]^{q/p}\Big)^{1/q}. \tag{2.35}$$

(b) The operators $S_{\tilde{\psi}} : \tilde{B}^s_{pq}\to\tilde{b}^s_{pq}$ *and* $T_\psi : \tilde{b}^s_{pq}\to\tilde{B}^s_{pq}$ *are bounded and* $T_\psi\circ S_{\tilde{\psi}} = Id$ *on* $\tilde{B}^s_{pq}$. *Hence,* $f\in\tilde{B}^s_{pq}\iff\{\langle f,\tilde{\psi}_\xi\rangle\}_{\xi\in\mathcal{X}}\in\tilde{b}^s_{pq}$. *Furthermore, if* $f\in\tilde{B}^s_{pq}$, *then* $\|f\|_{B^s_{pq}}\sim\|\{\langle f,\tilde{\psi}_\xi\rangle\}\|_{b^s_{pq}}$ *and under the reverse doubling condition* (1.7)

$$\|f\|_{\tilde{B}^s_{pq}} \sim \Big(\sum_{j\ge 0}\Big[\sum_{\xi\in\mathcal{X}_j}\Big(|B(\xi,b^{-j})|^{-s/d}\|\langle f,\tilde{\psi}_\xi\rangle\psi_\xi\|_p\Big)^p\Big]^{q/p}\Big)^{1/q}. \tag{2.36}$$

Theorem 2.11. [12] *Let* $s\in\mathbb{R}$, $0<p<\infty$ *and* $0<q\le\infty$. *(a) The operators* $S_{\tilde{\psi}} : F^s_{pq}\to f^s_{pq}$ *and* $T_\psi : f^s_{pq}\to F^s_{pq}$ *are bounded and* $T_{\tilde{\psi}}\circ S_\psi = Id$ *on* F^s_{pq}. *Consequently,* $f\in F^s_{pq}$ *if and only if* $\{\langle f,\tilde{\psi}_\xi\rangle\}_{\xi\in\mathcal{X}}\in f^s_{pq}$, *and if* $f\in F^s_{pq}$, *then* $\|f\|_{F^s_{pq}}\sim\|\{\langle f,\tilde{\psi}_\xi\rangle\}\|_{f^s_{pq}}$. *Furthermore,*

$$\|f\|_{F^s_{pq}} \sim \Big\|\Big(\sum_{j\ge 0} b^{jsq}\sum_{\xi\in\mathcal{X}_j}\big[|\langle f,\tilde{\psi}_\xi\rangle||\psi_\xi(\cdot)|\big]^q\Big)^{1/q}\Big\|_{L^p}. \tag{2.37}$$

(b) The operators $S_{\tilde{\psi}} : \tilde{F}^s_{pq} \to \tilde{f}^s_{pq}$ and $T_\psi : \tilde{f}^s_{pq} \to \tilde{F}^s_{pq}$ are bounded and $T_{\tilde{\psi}} \circ S_\psi = Id$ on $\tilde{F}^s_{pq}$. Hence, $f \in \tilde{F}^s_{pq}$ if and only if $\{\langle f, \tilde{\psi}_\xi\rangle\}_{\xi\in\mathcal{X}} \in \tilde{f}^s_{pq}$, and if $f \in F^s_{pq}$, then $\|f\|_{F^s_{pq}} \sim \|\{\langle f, \tilde{\psi}_\xi\rangle\}\|_{\tilde{f}^s_{pq}}$. Furthermore,

$$\|f\|_{\tilde{F}^s_{pq}} \sim \Big\|\Big(\sum_{\xi\in\mathcal{X}} \big[|B(\xi, b^{-j})|^{-s/d}|\langle f, \tilde{\psi}_\xi\rangle||\psi_\xi(\cdot)|\big]^q\Big)^{1/q}\Big\|_{L^p}. \tag{2.38}$$

The roles of $\{\psi_\xi\}$ and $\{\tilde{\psi}_\xi\}$ in Theorems 2.10-2.11 can be interchanged.

2.5. **Maximal operator.** We shall need the maximal operator $\mathcal{M}_t$ defined by

$$\mathcal{M}_t f(x) := \sup_{B\ni x}\left(\frac{1}{|B|}\int_B |f|^t\,d\mu\right)^{1/t}, \quad x \in M,\ t > 0, \tag{2.39}$$

where the sup is over all balls $B \subset M$ such that $x \in B$. Since μ is a Radon measure on M which satisfies the doubling condition (1.2) the Fefferman-Stein vector-valued maximal inequality holds (see [23]): If $0 < p < \infty, 0 < q \le \infty$, and $0 < t < \min\{p, q\}$ then for any sequence of functions $\{f_\nu\}$ on M

$$\Big\|\Big(\sum_\nu |\mathcal{M}_t f_\nu(\cdot)|^q\Big)^{1/q}\Big\|_{L^p} \le c_\sharp \Big\|\Big(\sum_\nu |f_\nu(\cdot)|^q\Big)^{1/q}\Big\|_{L^p}. \tag{2.40}$$

From Theorem 2.1 in [9] it follows that the constant $c_\sharp > 0$ above can be written in the form

$$c_\sharp = c_1 \max\big\{p, (p/t-1)^{-1}\big\}\max\big\{1, (q/t-1)^{-1}\big\}, \tag{2.41}$$

where c_1 is a structural constant depending only on the underlying space M.

3. General small perturbation method for construction of frames

The purpose of this section is to develop in general a small perturbation method for construction of frames in the case when there exists a pair of dual frames $\{\psi_\xi\}$, $\{\tilde{\psi}_\xi\}$ for a quasi-Banach space $\mathcal{B}$ of distributions (or a class Y of spaces $\mathcal{B}$).

3.1. **Assumptions in the case of a single space $\mathcal{B}$.** Assume that (M, ρ, μ) is a metric measure space and $\mathcal{D} \subset L^2(M, \mu)$ is a linear space of test functions on M furnished with a locally convex topology induced by a sequence of norms or semi-norms. Let $\mathcal{D}'$ be the dual of $\mathcal{D}$ consisting of all continuous linear functionals on $\mathcal{D}$. The pairing of $f \in \mathcal{D}'$ and $\phi \in \mathcal{D}$ will be denoted by $\langle f, \phi\rangle := f(\overline{\phi})$ and we assume that it is consistent with the inner product $\langle f, g\rangle = \int_M f\overline{g}d\mu$ on $L^2(M, \mu)$.

Further, we assume that $\mathcal{B} \subset \mathcal{D}'$ with norm $\|\cdot\|_{\mathcal{B}}$ is a quasi-Banach space of distributions, which is continuously embedded in $\mathcal{D}'$ and $\mathcal{D} \subset \mathcal{B}$.

The old pair of frames. We stipulate the existence of a pair of dual frames $\{\psi_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde{\psi}_\xi\}_{\xi\in\mathcal{X}}$ in $\mathcal{B}$, where $\psi_\xi, \tilde{\psi}_\xi \in \mathcal{D}$ and $\mathcal{X}$ is a countable index set, with the following properties:

A1. For any $f \in \mathcal{B}$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \tilde{\psi}_\xi\rangle\psi_\xi = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle\tilde{\psi}_\xi, \tag{3.1}$$

where the two series converge unconditionally in $\mathcal{B}$ and hence in $\mathcal{D}'$.

A2. Consider the following *analysis* and *synthesis* operators: $S_{\tilde{\psi}} : f \mapsto (\langle f, \tilde{\psi}_\xi\rangle)_{\xi\in\mathcal{X}}$ and $T_\psi : (h_\xi)_{\xi\in\mathcal{X}} \mapsto \sum_{\xi\in\mathcal{X}} h_\xi\psi_\xi$. The condition is that there exists a quasi-Banach (complex) sequence space $\mathcal{B}_d$ with quasi-norm $\|\cdot\|_{\mathcal{B}_d}$ such that:

(i) the operator $S_{\tilde{\psi}} : \mathcal{B} \mapsto \mathcal{B}_d$ is bounded, and

(ii) for any sequence $h = (h_\xi)_{\xi\in\mathcal{X}} \in \mathcal{B}_d$ the series $\sum_{\xi\in\mathcal{X}} h_\xi\psi_\xi$ converges unconditionally in $\mathcal{B}$ and $T_\psi : \mathcal{B}_d \to \mathcal{B}$ is bounded. Furthermore, the roles of ψ and $\tilde{\psi}$ can be interchanged.

Therefore, for any $f \in \mathcal{B}$ we have $(\langle f, \tilde{\psi}_\xi\rangle)_{\xi\in\mathcal{X}} \in \mathcal{B}_d$, $(\langle f, \psi_\xi\rangle)_{\xi\in\mathcal{X}} \in \mathcal{B}_d$, and $\|f\|_{\mathcal{B}} \sim \|(\langle f, \tilde{\psi}_\xi\rangle)\|_{\mathcal{B}_d} \sim \|(\langle f, \psi_\xi\rangle)\|_{\mathcal{B}_d}$.

In addition, we assume that $\mathcal{B}_d$ obeys the conditions:

A3. (i) For any sequence $(h_\xi)_{\xi\in\mathcal{X}} \in \mathcal{B}_d$, $\|(h_\xi)\|_{\mathcal{B}_d} = \|(|h_\xi|)\|_{\mathcal{B}_d}$.

(ii) Let $h = (h_\xi)_{\xi\in\mathcal{X}} \in \mathcal{B}_d$ and assume $(h_{\xi_j})_{j\ge1}$ is any ordering of the terms of the sequence h. Set $\mathcal{X}_m := \{\xi_j : j \ge m\}$ and define the *truncated* sequence $h^{(m)} \in \mathcal{B}_d$ by

$$h_\xi^{(m)} := h_\xi \ \text{ if } \ \xi \in \mathcal{X}_m \quad \text{and} \quad h_\xi^{(m)} := 0 \ \text{ if } \ \xi \in \mathcal{X} \setminus \mathcal{X}_m.$$

The condition is that $\|h^{(m)}\|_{\mathcal{B}_d} \to 0$ as $m \to \infty$.

Clearly, this assumption implies that compactly supported sequences are dense in $\mathcal{B}_d$.

A4. The operator with matrix

$$A := (a_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad a_{\xi,\eta} := \langle\psi_\eta, \psi_\xi\rangle \tag{3.2}$$

is bounded on $\mathcal{B}_d$, i.e. $\|A\|_{\mathcal{B}_d\mapsto\mathcal{B}_d} \le c < \infty$.

3.2. **Construction of new frames.** Next we construct a new pair of dual frames $\{\theta_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde{\theta}_\xi\}_{\xi\in\mathcal{X}}$ in $\mathcal{B}$, where $\mathcal{X}$ is the index set from above, in two steps: We first construct the new system $\{\theta_\xi\}_{\xi\in\mathcal{X}}$ to approximate well $\{\psi_\xi\}_{\xi\in\mathcal{X}}$ in terms of the size of the inner products $\langle\psi_\eta - \theta_\eta, \psi_\xi\rangle$ and be well localized in terms of $\langle\theta_\eta, \psi_\xi\rangle$. More precisely, we **assume** that $\theta_\xi \in \mathcal{B}$, $\xi \in \mathcal{X}$, can be constructed so that the operators with matrices

$$\begin{aligned} B &:= (b_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad b_{\xi,\eta} := \langle\theta_\eta, \psi_\xi\rangle, \\ D &:= (d_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad d_{\xi,\eta} := \langle\psi_\eta - \theta_\eta, \psi_\xi\rangle \end{aligned} \tag{3.3}$$

are bounded on $\mathcal{B}_d$, e.g. $\|B\|_{\mathcal{B}_d\mapsto\mathcal{B}_d} \le c$, and more importantly for a sufficiently small $\varepsilon > 0$ (to be determined later on)

$$\|D\|_{\mathcal{B}_d\mapsto\mathcal{B}_d} \le \varepsilon. \tag{3.4}$$

We introduce two operators:

$$T_d h := \sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi, \quad h = (h_\xi)_{\xi\in\mathcal{X}} \in \mathcal{B}_d, \ \text{ and} \tag{3.5}$$

$$Tf := \sum_{\xi\in\mathcal{X}} \langle f, \tilde{\psi}_\xi\rangle\theta_\xi, \quad f \in \mathcal{B}. \tag{3.6}$$

Lemma 3.1. *The operators T_d and T are well defined and bounded, that is,*

$$\|T_d h\|_{\mathcal{B}} \le c\|h\|_{\mathcal{B}_d}, \quad \forall h \in \mathcal{B}_d \quad \text{and} \quad \|Tf\|_{\mathcal{B}} \le c\|f\|_{\mathcal{B}}, \quad \forall f \in \mathcal{B}. \tag{3.7}$$

Furthermore, the series in (3.5)-(3.6) *converge unconditionally in $\mathcal{B}$ and hence in $\mathcal{D}'$.*

Proof. Let $h = (h_\xi)_{\xi\in\mathcal{X}}$ be a compactly supported sequence of complex numbers. Then

$$\langle T_d h, \psi_\eta\rangle = \sum_\xi h_\xi \langle\theta_\xi, \psi_\eta\rangle = (Bh)_\eta, \quad \eta \in \mathcal{X},$$

and using that $\|B\|_{\mathcal{B}_d\mapsto\mathcal{B}_d} \le c$ and **A2** we obtain

$$\|T_d h\|_{\mathcal{B}} \le c\|Bh\|_{\mathcal{B}_d} \le c\|B\|_{\mathcal{B}_d\mapsto\mathcal{B}_d}\|h\|_{\mathcal{B}_d} \le c'\|h\|_{\mathcal{B}_d}.$$

This and condition **A3**(ii) on $\mathcal{B}_d$ readily imply that the series in (3.5) converges unconditionally in $\mathcal{B}$ and T_d can be extended as a bounded operator from $\mathcal{B}_d$ to $\mathcal{B}$.

We use the above and **A2** to conclude that for any $f \in \mathcal{B}$

$$\|Tf\|_{\mathcal{B}} \le c\|(\langle f, \tilde\psi_\xi\rangle)\|_{\mathcal{B}_d} \le c\|f\|_{\mathcal{B}}. \quad \square$$

It will be critical that the operator T is invertible if ε in (3.4) is sufficiently small.

Lemma 3.2. *If ε in* (3.4) *is sufficiently small and independent of other constants, then $\|I - T\|_{\mathcal{B}\mapsto\mathcal{B}} < 1$ and hence T^{-1} exists and*

$$\|T^{-1}\|_{\mathcal{B}\mapsto\mathcal{B}} \le c < \infty. \tag{3.8}$$

Proof. For $f \in \mathcal{B}$ we have (with I being the identity operator)

$$(I - T)f = \sum_{\xi\in\mathcal{X}} \langle f, \tilde\psi_\xi\rangle(\psi_\xi - \theta_\xi),$$

where the convergence in $\mathcal{B}$ and hence in $\mathcal{D}'$. Therefore,

$$\langle (I - T)f, \psi_\eta\rangle = \sum_{\xi\in\mathcal{X}} \langle f, \tilde\psi_\xi\rangle\langle\psi_\xi - \theta_\xi, \psi_\eta\rangle = (Dh)_\eta,$$

where D is from (3.3) and $h_\xi := \langle f, \tilde\psi_\xi\rangle$. Now, using **A2** and (3.4) we obtain

$$\|(I - T)f\|_{\mathcal{B}} \le c\|Dh\|_{\mathcal{B}_d} \le c\|D\|_{\mathcal{B}_d\mapsto\mathcal{B}_d}\|h\|_{\mathcal{B}_d} \le c\varepsilon\|h\|_{\mathcal{B}_d} \le c_*\varepsilon\|f\|_{\mathcal{B}}.$$

Hence $\|I - T\|_{\mathcal{B}\mapsto\mathcal{B}} \le c_*\varepsilon < 1$ if ε is sufficiently small.

By our hypotheses $\mathcal{B}$ is a quasi-Banach space and as is well known there exists a constant $0 < p \le 1$ such that $\|\sum_j f_j\|_{\mathcal{B}}^p \le \sum_j \|f_j\|_{\mathcal{B}}^p$ for $f_j \in \mathcal{B}$. Using this it is easy to show that $\|I-T\|_{\mathcal{B}\mapsto\mathcal{B}} < 1$ implies that T^{-1} exists and $\|T^{-1}\|_{\mathcal{B}\mapsto\mathcal{B}} \le c < \infty$. In fact, $T^{-1} = \sum_{k\ge0}(I-T)^k$ and $\|T^{-1}\|^p \le \sum_{k\ge0}\|I-T\|^{pk} \le (1-(c_*\varepsilon)^p)^{-1} < \infty$. $\square$

We need one more simple lemma:

Lemma 3.3. *The operators H with matrix $H := (\langle T^{-1}\psi_\eta, \tilde\psi_\xi\rangle)_{\xi,\eta\in\mathcal{X}}$ is bounded on $\mathcal{B}_d$.*

Proof. Let $h = (h_\xi)_{\xi\in\mathcal{X}}$ be a compactly supported sequence of complex numbers and set $f := \sum_{\eta\in\mathcal{X}} h_\xi\psi_\eta$. Clearly,

$$(Hh)_\xi = \sum_{\eta\in\mathcal{X}} h_\eta\langle T^{-1}\psi_\eta, \tilde\psi_\xi\rangle = \Big\langle T^{-1}\Big(\sum_{\eta\in\mathcal{X}} h_\eta\psi_\eta\Big), \tilde\psi_\xi\Big\rangle = \langle T^{-1}f, \tilde\psi_\xi\rangle.$$

The above, **A2**, and (3.8) imply

$$\|Hh\|_{\mathcal{B}_d} = \|(\langle T^{-1}f, \tilde\psi_\xi\rangle)\|_{\mathcal{B}_d} \le c\|T^{-1}f\|_{\mathcal{B}} \le c\|f\|_{\mathcal{B}} \le c\|h\|_{\mathcal{B}_d}.$$

Since compactly supported sequences are dense in $\mathcal{B}_d$ the operator H can be uniquely extended to a bounded operator on $\mathcal{B}_d$. $\square$

Construction of the dual frame $\{\tilde\theta_\xi\}$. For any $\xi \in \mathcal{X}$ we define the linear functional $\tilde\theta_\xi$ by

$$\tilde\theta_\xi(f) = \langle f, \tilde\theta_\xi\rangle := \sum_{\eta\in\mathcal{X}} \langle T^{-1}\psi_\eta, \tilde\psi_\xi\rangle\langle f, \tilde\psi_\eta\rangle \quad \text{for } f \in \mathcal{B}. \tag{3.9}$$

From Lemma 3.3 and **A2** it follows that for $f \in \mathcal{B}$

$$|\tilde\theta_\xi(f)| = |\langle f, \tilde\theta_\xi\rangle| \le \|H\|_{\mathcal{B}_d\mapsto\mathcal{B}_d}\|(\langle f, \tilde\psi_\eta\rangle)\|_{\mathcal{B}_d} \le c\|f\|_{\mathcal{B}}.$$

Thus, $\tilde\theta_\xi$ is a bounded linear functional on $\mathcal{B}$, i.e. $\tilde\theta_\xi \in \mathcal{B}'$.

In going further, for $f \in \mathcal{B}$ by Lemma 3.2 $T^{-1}f \in \mathcal{B}$ and using Lemma 3.1

$$f = T(T^{-1}f) = \sum_{\xi\in\mathcal{X}} \langle T^{-1}f, \tilde\psi_\xi\rangle\theta_\xi. \tag{3.10}$$

On the other hand, from the fact that T^{-1} is a bounded operator on $\mathcal{B}$ and (3.1) it follows that for any $f \in \mathcal{B}$ we have $T^{-1}f = \sum_{\eta\in\mathcal{X}}\langle f, \tilde\psi_\eta\rangle T^{-1}\psi_\eta$, where the series converges unconditionally in $\mathcal{B}$ and hence in $\mathcal{D}'$. This and the fact that $\tilde\psi_\xi \in \mathcal{D}$ imply

$$\langle T^{-1}f, \tilde\psi_\xi\rangle = \sum_{\eta\in\mathcal{X}} \langle T^{-1}\psi_\eta, \tilde\psi_\xi\rangle\langle f, \tilde\psi_\eta\rangle = \langle f, \tilde\theta_\xi\rangle. \tag{3.11}$$

Here the series converges unconditionally and hence absolutely because of the unconditional convergence of the former series. From (3.10)-(3.11) we infer

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \tilde\theta_\xi\rangle\theta_\xi, \quad f \in \mathcal{B}, \tag{3.12}$$

where $\langle f, \tilde\theta_\xi\rangle$ is defined in (3.9); the convergence is unconditional in $\mathcal{B}$.

We next show that $\tilde\theta_\xi$ can be identified in a sense with an element of $\mathcal{B}$.

Proposition 3.4. *For any $\xi \in \mathcal{X}$ the distribution*

$$\tilde\theta_\xi := \sum_{\eta\in\mathcal{X}} \overline{\langle T^{-1}\psi_\eta, \tilde\psi_\xi\rangle}\tilde\psi_\eta \quad (\textit{convergence in } \mathcal{B}) \tag{3.13}$$

belongs to $\mathcal{B}$ and for any $\phi \in \mathcal{D}$ we have

$$\tilde\theta_\xi(\phi) = \overline{\langle \tilde\theta_\xi, \phi\rangle}, \tag{3.14}$$

where on the left the linear functional $\tilde\theta_\xi \in \mathcal{B}'$, defined in (3.9)*, acts on $\phi \in \mathcal{D} \subset \mathcal{B}$, while on the right the distribution $\tilde\theta_\xi \in \mathcal{B}$ from* (3.13) *acts on $\phi \in \mathcal{D}$.*

Proof. Assume for a moment that the series in (3.13) converges in $\mathcal{B}$ and hence in $\mathcal{D}'$. Then we have for $\phi \in \mathcal{D}$

$$\overline{\langle \tilde\theta_\xi, \phi\rangle} = \overline{\Big\langle \sum_{\xi\in\mathcal{X}} \overline{\langle T^{-1}\psi_\eta, \tilde\psi_\xi\rangle}\tilde\psi_\eta, \phi\Big\rangle} = \sum_{\xi\in\mathcal{X}} \langle T^{-1}\psi_\eta, \tilde\psi_\xi\rangle\langle\phi, \tilde\psi_\eta\rangle = \tilde\theta_\xi(\phi),$$

where for the last equality we used (3.9); this verifies (3.14).

To show that the series in (3.13) converges in $\mathcal{B}$, we observe that as $\psi_\eta \in \mathcal{D} \subset \mathcal{B}$ by (3.1) $\psi_\eta = \sum_{\omega\in\mathcal{X}}\langle \psi_\eta, \tilde\psi_\omega\rangle\psi_\omega$ $\forall\eta \in \mathcal{X}$ with convergence in $\mathcal{B}$. Hence, using that the operator T^{-1} is bounded on $\mathcal{B}$ it follows that $T^{-1}\psi_\eta = \sum_{\omega\in\mathcal{X}}\langle\psi_\eta, \tilde\psi_\omega\rangle T^{-1}\psi_\omega$ in $\mathcal{B}$ and hence in $\mathcal{D}'$. Therefore, as $\tilde\psi_\xi \in \mathcal{D}$

$$\langle T^{-1}\psi_\eta, \tilde\psi_\xi\rangle = \sum_{\omega\in\mathcal{X}} \langle T^{-1}\psi_\omega, \tilde\psi_\xi\rangle\langle\psi_\eta, \tilde\psi_\omega\rangle.$$

However, by Lemma 3.3 the operator H with matrix $H := (\langle T^{-1}\psi_\omega, \tilde{\psi}_\xi\rangle)_{\xi,\omega\in\mathcal{X}}$ is bounded on $\mathcal{B}_d$, and $\big(\langle \psi_\eta, \tilde{\psi}_\omega\rangle\big)_{\eta\in\mathcal{X}} = \big(\overline{\langle \tilde{\psi}_\omega, \psi_\eta\rangle}\big)_{\eta\in\mathcal{X}} \in \mathcal{B}_d$ since $\tilde{\psi}_\omega \in \mathcal{D} \subset \mathcal{B}$ and using **A2-A3**. Therefore, $\big(\langle T^{-1}\psi_\eta, \tilde{\psi}_\xi\rangle\big)_{\eta\in\mathcal{X}} \in \mathcal{B}_d$ and using **A3** we have $\big(\overline{\langle T^{-1}\psi_\eta, \tilde{\psi}_\xi\rangle}\big)_{\eta\in\mathcal{X}} \in \mathcal{B}_d$. Then from **A2** it follows that the series in (3.13) converges in $\mathcal{B}$ and $\tilde{\theta}_\xi \in \mathcal{B}$. $\square$

We next show that in a sense $\{\theta_\xi\}$, $\{\tilde{\theta}_\xi\}$ is a pair of dual frames for $\mathcal{B}$ if ε is sufficiently small.

Theorem 3.5. *If ε in* (3.4) *is sufficiently small, with the above defined $\{\theta_\xi\}$, $\{\tilde{\theta}_\xi\}$, for any $f \in \mathcal{B}$*

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \tilde{\theta}_\xi\rangle \theta_\xi, \tag{3.15}$$

where the convergence is unconditional in $\mathcal{B}$, and

$$\|f\|_{\mathcal{B}} \sim \|(\langle f, \tilde{\theta}_\xi\rangle)\|_{\mathcal{B}_d} \tag{3.16}$$

with constants of equivalence independent of f.

Moreover, the operator T_d defined by $T_d h := \sum_{\xi\in\mathcal{X}} h_\xi\theta_\xi$ for sequences of complex numbers $h = (h_\xi)_{\xi\in\mathcal{X}}$ is bounded as a map $T_d : \mathcal{B}_d \mapsto \mathcal{B}$.

Proof. Representation (3.15) was already established in (3.12). To prove that

$$\|f\|_{\mathcal{B}} \le c\|(\langle f, \tilde{\theta}_\xi\rangle)\|_{\mathcal{B}_d}, \quad f \in \mathcal{B}, \tag{3.17}$$

we first use **A2** to obtain $\|f\|_{\mathcal{B}} \le c\|(\langle f, \tilde{\psi}_\xi\rangle)\|_{\mathcal{B}_d}$. Using (3.11) we write

$$\langle f, \tilde{\psi}_\xi\rangle = \langle f - T^{-1}f, \tilde{\psi}_\xi\rangle + \langle T^{-1}f, \tilde{\psi}_\xi\rangle = \langle T^{-1}(I-T)f, \tilde{\psi}_\xi\rangle + \langle f, \tilde{\theta}_\xi\rangle.$$

Now from **A2**, (3.7), (3.8), and $\|I - T\|_{\mathcal{B}\mapsto\mathcal{B}} \le c_*\varepsilon$ (Lemma 3.2) it follows that

$$\begin{aligned}\|f\|_{\mathcal{B}} &\le c\|(\langle f, \tilde{\psi}_\xi\rangle)\|_{\mathcal{B}_d} \le c\|(\langle T^{-1}(T-I)f, \tilde{\psi}_\xi\rangle)\|_{\mathcal{B}_d} + c\|(\langle f, \tilde{\theta}_\xi\rangle)\|_{\mathcal{B}_d}\\ &\le c\|T^{-1}(T-I)f\|_{\mathcal{B}} + c\|(\langle f, \tilde{\theta}_\xi\rangle)\|_{\mathcal{B}_d}\\ &\le c\|T^{-1}\|_{\mathcal{B}\mapsto\mathcal{B}}\|T-I\|_{\mathcal{B}\mapsto\mathcal{B}}\|f\|_{\mathcal{B}} + c\|(\langle f, \tilde{\theta}_\xi\rangle)\|_{\mathcal{B}_d} \le c_\diamond\varepsilon\|f\|_{\mathcal{B}} + c\|(\langle f, \tilde{\theta}_\xi\rangle)\|_{\mathcal{B}_d}\end{aligned}$$

with $c_\diamond > 0$ a constant independent of ε. Therefore,

$$\|f\|_{\mathcal{B}} \le \frac{c}{1 - c_\diamond\varepsilon}\|(\langle f, \tilde{\theta}_\xi\rangle)\|_{\mathcal{B}_d},$$

which implies (3.17) if we choose ε so that $c_\diamond\varepsilon < 1$.

In the other direction, we use (3.11), **A2**, and (3.8) to obtain

$$\|(\langle f, \tilde{\theta}_\xi\rangle)\|_{\mathcal{B}_d} = \|(\langle T^{-1}f, \tilde{\psi}_\xi\rangle)\|_{\mathcal{B}_d} \le c\|T^{-1}f\|_{\mathcal{B}} \le c\|f\|_{\mathcal{B}}.$$

The boundedness of $T_d : \mathcal{B}_d \mapsto \mathcal{B}$ is established in Lemma 3.1. $\square$

3.3. **Construction of frames in the case of existence of a single frame.** There are many cases when there is a single (old) frame $\{\psi_\xi\}$ for a quasi-Banach space $\mathcal{B}$. More specifically, assume that in the setting of §3.1 $\tilde{\psi}_\xi = \psi_\xi$, $\xi \in \mathcal{X}$, i.e. for any $f \in \mathcal{B}$

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \psi_\xi\rangle\psi_\xi \quad \text{and} \quad \|f\|_{\mathcal{B}} \sim \|(\langle f, \psi_\xi\rangle)\|_{\mathcal{B}_d}. \tag{3.18}$$

In this situation the construction of a new pair of frames $\{\theta_\xi\}$, $\{\tilde\theta_\xi\}$ can be simplified. More precisely, $\{\theta_\xi\}$ is constructed as in §3.2 and $\{\tilde\theta_\xi\}$ is defined by $\tilde\theta_\xi := S^{-1}\theta_\xi$, $\xi \in \mathcal{X}$, where S is the frame operator: $Sf := \sum_{\xi\in\mathcal{X}}\langle f, \theta_\xi\rangle\theta_\xi$. This method is developed in [16], where it is shown that if the the operators with matrices B, D from (3.3) and their adjoints B^*, D^* are bounded on $\mathcal{B}_d$ and ℓ^2, and for a sufficiently small ε, $\|D\|_{\mathcal{B}_d\to\mathcal{B}_d} \le \varepsilon$, $\|D^*\|_{\mathcal{B}_d\to\mathcal{B}_d} \le \varepsilon$, $\|D\|_{\ell^2\to\ell^2} \le \varepsilon$, then S^{-1} exists and is bounded on $\mathcal{B}$ and for any $f \in \mathcal{B}$

$$f = \sum_{\xi\in\mathcal{X}}\langle f, \tilde\theta_\xi\rangle\theta_\xi \quad \text{and} \quad \|f\|_{\mathcal{B}} \sim \|(\langle f, \tilde\theta_\xi\rangle)\|_{\mathcal{B}_d}. \tag{3.19}$$

We refer the reader to [16] for details and proofs.

3.4. **Construction of frames for classes of quasi-Banach spaces.** Let Y be a class (set) of quasi-Banach spaces $\mathcal{B}$ of distributions and assume that $\{\psi_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde\psi_\xi\}_{\xi\in\mathcal{X}}$ is a pair of dual frames, just as in §3.1, for each $\mathcal{B} \in Y$. We shall denote by Y_d the class consisting of the respective sequence spaces $\mathcal{B}_d$.

Now, our *main assumption* is that all constants in §3.1 are uniform with respect to $\mathcal{B} \in Y$ and $\mathcal{B}_d \in Y_d$, i.e. they are the same for all $\mathcal{B} \in Y$ and $\mathcal{B}_d \in Y_d$.

In the construction in §3.2 of new frames $\{\theta_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde\theta_\xi\}_{\xi\in\mathcal{X}}$ for $\mathcal{B} \in Y$ our main assumption now is that the constants are also uniform. Thus we assume that $\theta_\xi \in \mathcal{D}$, $\xi \in \mathcal{X}$, can be constructed so that $\|B\|_{\mathcal{B}_d\mapsto\mathcal{B}_d} \le c$ and $\|D\|_{\mathcal{B}_d\mapsto\mathcal{B}_d} \le \varepsilon$ for all $\mathcal{B}_d \in Y_d$, where $\varepsilon > 0$ is sufficiently small.

A careful examination of the arguments shows that under the above assumptions Theorem 3.5 holds for all $\mathcal{B} \in Y$, where the constants in (3.16) are independent of $\mathcal{B}$ as well; they may depend on Y, Y_d, and the constants from the assumptions.

4. Compactly supported frames in Dirichlet spaces

In this section we present the construction of a compactly supported frame $\{\theta_\xi\}$ in the general setting of §1 and its dual frame $\{\tilde\theta_\xi\}$, and show how they can be used for characterization of Besov and Triebel-lizorkin spaces.

4.1. **The construction.** Let $\Psi_0 := \Phi$ and Ψ be the compactly supported C^∞ functions from the construction of Frame # 1 in §2.3. The first step is to construct band limited functions Θ_0 and Θ which approximate well Ψ_0 and Ψ in a specific sense given below.

Proposition 4.1. *Let Ψ_0 and Ψ be the even extensions of the functions Ψ_0 and Ψ from the construction of Frame # 1 in §2.3. Then for any $\varepsilon > 0$ and $N \ge K \ge 1$ there exist functions $\Theta_0, \Theta \in C^\infty$ and $R > 0$ such that Θ_0 and Θ are even and real valued,* $\operatorname{supp}\hat\Theta_0 \subset [-R, R]$, $\operatorname{supp}\hat\Theta \subset [-R, R]$,

$$|\Psi_0^{(\nu)}(t) - \Theta_0^{(\nu)}(t)| \le \frac{\varepsilon|t|^N}{(1+|t|)^{2N}}, \quad t \in \mathbb{R}, \quad \nu = 0, 1, \dots, K, \tag{4.1}$$

and

$$|\Psi^{(\nu)}(t) - \Theta^{(\nu)}(t)| \le \frac{\varepsilon|t|^N}{(1+|t|)^{2N}}, \quad t \in \mathbb{R}, \quad \nu = 0, 1, \dots, K. \tag{4.2}$$

Furthermore, $\operatorname{supp}\mathcal{F}(t^{-m}\Theta(t)) \subset [-R, R]$*, $0 \le m \le N$, with $\mathcal{F}$ being the Fourier transform.*

Proof. For this proof we shall borrow from [12] and [16]. Evidently, it suffices to prove the proposition in the case when $N = K = k > 1$.

We first construct Θ. Define $f(t) := (\sin\gamma t)^{-2k}\Psi(t)$ with $\gamma := \pi/2b$, and observe that $f \in C^\infty(\mathbb{R})$, f is even, and $\operatorname{supp} f = \operatorname{supp}\Psi \subset [-b, b]$, $b \ge 2$.

Our next step is to construct a band limited function f_A, $A > 1$, which approximates well f. To this end we shall proceed similarly as in the proof of Theorem 3.1 in [12]. Just as in [12] we define the function ϕ on $\mathbb{R}$ by its Fourier transform

$$\hat\phi := \mathbb{1}_{[-\frac12-\delta,\frac12+\delta]} * \underbrace{H_\delta * \cdots * H_\delta}_{k+1}, \quad \text{where } H_\delta := (2\delta)^{-1}\mathbb{1}_{[-\delta,\delta]}, \quad \delta := \frac{1}{2(k+2)}.$$

Evidently, $\hat\phi$ is even, $\operatorname{supp}\hat\phi \subset [-1, 1]$, $0 \le \hat\phi \le 1$, $\hat\phi(\xi) = 1$ for $\xi \in [-1/2, 1/2]$, and

$$\|\hat\phi^{(\nu)}\|_\infty \le \delta^{-\nu} \le (2(k+2))^\nu \le (4k)^\nu \quad \text{for } \nu = 0, 1, \dots, k+1.$$

Define $f_A := f * \phi_A$, where $\phi_A(t) := A\phi(At)$. Note that $\widehat{\phi_A}(\xi) = \hat\phi(\xi/A)$ and hence $\operatorname{supp}\widehat{\phi_A} \subset [-A, A]$. On the other hand, $\widehat{f_A} = \hat f\widehat{\phi_A}$ and, therefore, $\operatorname{supp}\widehat{f_A} \subset [-A, A]$. Since f and ϕ are even, f_A is even. In going further,

$$f(t) - f_A(t) = (2\pi)^{-1}A^{-k}\int_{\mathbb{R}} \xi^k \hat f(\xi)\hat F(\xi/A)e^{i\xi t}d\xi,$$

where $\hat F(\xi) = (1 - \hat\phi(\xi))\xi^{-k}$. From this we infer

$$f^{(\nu)}(t) - f_A^{(\nu)}(t) = i^\nu(2\pi)^{-1}A^{-k}\int_{\mathbb{R}} \xi^{k+\nu}\hat f(\xi)\hat F(\xi/A)e^{i\xi t}d\xi$$

and hence

$$\begin{aligned}\|f^{(\nu)} - f_A^{(\nu)}\|_\infty &\le A^{-k}\|f^{(k+\nu)} * F_A\|_\infty \\ &\le A^{-k}\|f^{(k+\nu)}\|_\infty\|F_A\|_{L^1} \le c^kA^{-k}\|f^{(k+\nu)}\|_\infty.\end{aligned} \tag{4.3}$$

Here we used that $\|F_A\|_{L^1} = \|F\|_{L^1} \le c^k$, where $c > 1$ is an absolute constant [12].

As in [12] we have

$$|\phi_A(t)| \le c(k)A(1 + A|t|)^{-k-1}, \quad c(k) = (c'k)^k.$$

We use this and $\operatorname{supp} f \subset [-b, b]$ to obtain for $t > b$

$$\begin{aligned}|f^{(\nu)}(t) - f_A^{(\nu)}(t)| = |f_A^{(\nu)}(t)| = |f^{(\nu)} * \phi_A(t)| &\le \int_{-b}^{b} |f^{(\nu)}(y)||\phi_A(y-t)|dy \\ &\le \|f^{(\nu)}\|_\infty\int_{t-b}^{t+b}|\phi_A(u)|du \le c(k)\|f^{(\nu)}\|_\infty\int_{t-b}^{t+b}A(1+Au)^{-k-1}du \\ &\le c(k)\|f^{(\nu)}\|_\infty\int_{A(t-b)}^{\infty}(1+v)^{-k-1}du \le c(k)A^{-k}\|f^{(\nu)}\|_\infty(t-b)^{-k}.\end{aligned}$$

Therefore,

$$|f^{(\nu)}(t) - f_A^{(\nu)}(t)| \le c'A^{-k}\|f^{(\nu)}\|_\infty(1 + |t|)^{-k} \quad \text{for } |t| \ge 2b.$$

This coupled with (4.3) yields

$$|f^{(\nu)}(t) - f_A^{(\nu)}(t)| \le cA^{-k}\max_{0\le j\le 2k}\|f^{(j)}\|_\infty(1 + |t|)^{-k} \le c'A^{-k}(1 + |t|)^{-k} \tag{4.4}$$

for $t \in \mathbb{R}$ and $\nu = 0, 1, \dots, k$, where $c' > 0$ is independent of t and A.

We set

$$\Theta(t) := (\sin\gamma t)^{2k}f_A(t) \quad \text{with } \gamma := \pi/2b \text{ as above.}$$

We next show that Θ and $t^{-m}\Theta(t)$ $(1 \le m \le 2k)$ are band limited. Indeed, set $\Delta_\gamma^{2k} := (T_\gamma - T_{-\gamma})^{2k}$, where T_γ is the shift to the left operator, defined by $T_\gamma g(\xi) := g(\xi+\gamma)$. It is readily seen that

$$\big(\Delta_\gamma^{2k}\widehat{f_A}\big)^\vee(t) = (-1)^k 2^{2k}(\sin\gamma t)^{2k} f_A(t) = (-1)^k 2^{2k}\Theta(t)$$

and hence

$$\hat\Theta(\xi) = (-1)^k 2^{-2k}\Delta_\gamma^{2k}\widehat{f_A}(\xi).$$

Since $\operatorname{supp}\widehat{f_A} \subset [-A, A]$, it follows that $\operatorname{supp}\hat\Theta \subset [-A-2k\gamma, A+2k\gamma]$.

In going further, set $G_\nu(t) := (\sin\gamma t)^{2k-2\nu} f_A(t)$, $0 \le \nu \le k$. Then

$$t^{-2\nu}\Theta(t) = (\sin\gamma t/t)^{2\nu} G_\nu(t).$$

As above $\operatorname{supp}\widehat{G}_\nu \subset [-A-2(k-\nu)\gamma, A+2(k-\nu)\gamma]$. Clearly, $\mathcal{F}(\sin\gamma t/t) = \pi\mathbb{1}_{[-\gamma,\gamma]}$ and hence

$$\mathcal{F}\big(t^{-2\nu}\Theta(t)\big) = (-1)^\nu\pi^{2\nu}\underbrace{\mathbb{1}_{[-\gamma,\gamma]} * \cdots * \mathbb{1}_{[-\gamma,\gamma]}}_{2\nu} * \widehat{G}_\nu.$$

Therefore, $\operatorname{supp}\mathcal{F}\big(t^{-2\nu}\Theta(t)\big) \subset [-A-2k\gamma, A+2k\gamma]$, $0 \le \nu \le k$. This along with the obvious fact that $\operatorname{supp}\mathcal{F}(tf(t)) = \operatorname{supp}\mathcal{F}(f')$ yields

$$\operatorname{supp}\mathcal{F}\big(t^{-m}\Theta(t)\big) \subset [-A-2k\gamma, A+2k\gamma] =: [-R, R], \quad 0 \le m \le 2k,$$

as claimed.

We now establish (4.2). From the definition of f and Θ

$$\Psi(t) - \Theta(t) := (\sin\gamma t)^{2k}[f(t) - f_A(t)]$$

and using (4.4)

$$|\Psi^{(\nu)}(t) - \Theta^{(\nu)}(t)| \le c|\sin\gamma t|^k \max_{0\le j\le\nu}|f^{(j)}(t) - f_A^{(j)}(t)| \le \frac{c' A^{-k}|t|^k}{(1+|t|)^{2k}},$$

for $\nu = 0, 1, \dots, k$.

Finally, choosing A so that $c'A^{-k} = \varepsilon$ and setting $R := A + 2k\gamma$ we get Θ with the claimed properties.

To construct Θ_0, we first note that $\Psi_0' \in C^\infty$, $\operatorname{supp}\Psi_0' \subset [-b, -b^{-1}] \cup [b^{-1}, b]$, and Ψ_0' is odd. Then just as above we construct an odd function Θ_0' which approximate Ψ_0' as above and $\operatorname{supp}\widehat{\Theta_0'} \subset [-R, R]$. Finally, we set $\Theta_0(t) := 1 + \int_0^t \Theta_0'(u)du$. It is easy to see that this will give us Θ_0 with the claimed properties. $\square$

Construction of compactly supported frame. The constants N, K, and ε (sufficiently small) will be selected later on. With these constants fixed, we use the functions Θ_0, Θ from Proposition 4.1 to define the new frame. Similarly as in (2.14) we set

$$\Theta_j(u) := \Theta(b^{-j}u), \quad j \ge 1. \tag{4.5}$$

Let the sets $\mathcal{X}_j$, $\{A_\xi\}_{\xi\in\mathcal{X}_j}$, and $\mathcal{X} := \cup_{j\ge0}\mathcal{X}_j$ be from the definition of Frame # 1 in §2.3. We define the jth level $(j \ge 0)$ elements of the new system by

$$\theta_\xi(x) := |A_\xi|^{1/2}\Theta_j(\sqrt{L})(x, \xi), \quad \xi \in \mathcal{X}_j. \tag{4.6}$$

Then $\{\theta_\xi\}_{\xi\in\mathcal{X}}$ is the new Frame #1. A dual frame $\{\tilde\theta_\xi\}_{\xi\in\mathcal{X}}$ is produced using the general scheme from §3.

Observe immediately that since $\operatorname{supp} \hat{\Theta}_0 \subset [-R, R]$ and $\operatorname{supp} \hat{\Theta} \subset [-R, R]$, by Proposition 2.1 it follows that each θ_ξ is **compactly supported**, more precisely

$$\operatorname{supp} \theta_\xi \subset B(\xi, \tilde{c}Rb^{-j}), \quad \xi \in \mathcal{X}_j,\ j \geq 0. \tag{4.7}$$

We shall assume that $\tilde{c}, R \geq 1$.

4.2. **Main result.** Our goal is to show that the above defined system $\{\theta_\xi\}_{\xi \in \mathcal{X}}$ along with a dual system $\{\tilde{\theta}_\xi\}_{\xi \in \mathcal{X}}$ constructed by the recipe from §3 form a pair of frames for the Besov and Triebel-Lizorkin spaces B^s_{pq}, $\tilde{B}^s_{pq}$, F^s_{pq}, and $\tilde{F}^s_{pq}$ defined in §2.4 for the following range of indices determined by constants $s_0 \geq 0$, $p_0, p_1, q_0 > 0$:

$$\Omega := \{(s, p, q) : |s| \leq s_0,\ p_0 \leq p \leq p_1, \text{ and } q_0 \leq q < \infty\}. \tag{4.8}$$

To state the result we also introduce the constant: $\mathcal{J}_0 := d/\min\{1, p_0\}$ in the case of B-spaces and $\mathcal{J}_0 := d/\min\{1, p_0, q_0\}$ in the case of F-spaces.

Theorem 4.2. *Suppose $s_0 \geq 0$, $p_0, p_1, q_0 > 0$, $p_1 \geq p_0$, and let $\{\theta_\xi\}_{\xi \in \mathcal{X}}$ be the system constructed in* (4.6)*, where*

$$K \geq s_0 + \mathcal{J}_0 + d/2 + 1 \quad \text{and} \quad N \geq K + s_0 + \mathcal{J}_0 + 3d/2 + 1.$$

If ε in the construction of $\{\theta_\xi\}_{\xi \in \mathcal{X}}$ is sufficiently small the following holds true for $(s, p, q) \in \Omega$ with Ω from (4.8)*:* (*a*) *The operator*

$$Tf := \sum_{\xi \in \mathcal{X}} \langle f, \tilde{\psi}_\xi \rangle \theta_\xi, \tag{4.9}$$

is invertible on B^s_{pq} and T, T^{-1} are bounded on B^s_{pq}, uniformly with respect to $(s, p, q) \in \Omega$.

(*b*) *The system $\{\tilde{\theta}_\xi\}_{\xi \in \mathcal{X}}$ consists of bounded linear functionals on B^s_{pq} defined by*

$$\tilde{\theta}_\xi(f) = \langle f, \tilde{\theta}_\xi \rangle := \sum_{\eta \in \mathcal{X}} \langle T^{-1}\psi_\eta, \tilde{\psi}_\xi \rangle \langle f, \tilde{\psi}_\eta \rangle \quad \text{for } f \in B^s_{pq}, \tag{4.10}$$

with the series converging absolutely, and $\tilde{\theta}_\xi$, $\xi \in \mathcal{X}$, can be identified with

$$\tilde{\theta}_\xi := \sum_{\eta \in \mathcal{X}} \overline{\langle T^{-1}\psi_\eta, \tilde{\psi}_\xi \rangle} \tilde{\psi}_\eta, \quad \text{where} \quad |\tilde{\theta}_\xi(x)| \leq \frac{c|B(\xi, b^{-j})|^{-1/2}}{\big(1 + b^j \rho(x, \xi)\big)^\sigma}, \quad x \in M, \tag{4.11}$$

in the sense that for any $\phi \in \mathcal{D}$ we have $\tilde{\theta}_\xi(\phi) = \langle \phi, \tilde{\theta}_\xi \rangle$ (*inner product*)*. Here $\sigma > 0$ is arbitrary but fixed.*

Moreover, $\{\theta_\xi\}_{\xi \in \mathcal{X}}$, $\{\tilde{\theta}_\xi\}_{\xi \in \mathcal{X}}$ form a pair of dual frames for B^s_{pq} in the following sense: For any $f \in B^s_{pq}$

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \tilde{\theta}_\xi \rangle \theta_\xi \quad \text{and} \quad \|f\|_{B^s_{pq}} \sim \|(\langle f, \tilde{\theta}_\xi \rangle)\|_{b^s_{pq}}, \tag{4.12}$$

where the convergence is unconditional in B^s_{pq}.

(*c*) *The operator T_d defined by $T_d h := \sum_{\xi \in \mathcal{X}} h_\xi \theta_\xi$ for sequences of numbers $h = (h_\xi)_{\xi \in \mathcal{X}}$ is bounded as a map $T_d : b^s_{pq} \mapsto B^s_{pq}$, uniformly relative to $(s, p, q) \in \Omega$.*

Furthermore, $(a) - (c)$ *above hold true when B^s_{pq} is replaced by $\tilde{B}^s_{pq}$, F^s_{pq}, or $\tilde{F}^s_{pq}$, and b^s_{pq} by $\tilde{b}^s_{pq}$, f^s_{pq}, or $\tilde{f}^s_{pq}$, respectively.*

4.3. **Almost diagonal matrices.** On account of Theorem 3.5 and the discussion in §3.4 it is clear that to show that $\{\theta_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde\theta_\xi\}_{\xi\in\mathcal{X}}$ is a pair of frames for the B- and F-spaces B^s_{pq}, $\tilde B^s_{pq}$, F^s_{pq}, and $\tilde F^s_{pq}$ for $(s,p,q)\in\Omega$ (see (4.8)) it suffices to show that the operators with matrices

$$\begin{aligned} A &:= (a_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad a_{\xi,\eta} := \langle\psi_\eta,\psi_\xi\rangle,\\ B &:= (b_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad b_{\xi,\eta} := \langle\theta_\eta,\psi_\xi\rangle,\\ D &:= (d_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad d_{\xi,\eta} := \langle\psi_\eta-\theta_\eta,\psi_\xi\rangle \end{aligned} \tag{4.13}$$

are bounded on the respective sequence spaces b^s_{pq}, $\tilde b^s_{pq}$, f^s_{pq}, and $\tilde f^s_{pq}$, defined by Definitions 2.8-2.9, and

$$\|D\|_{b^s_{pq}\mapsto b^s_{pq}}\le\varepsilon,\ \ \|D\|_{\tilde b^s_{pq}\mapsto\tilde b^s_{pq}}\le\varepsilon,\ \ \|D\|_{f^s_{pq}\mapsto f^s_{pq}}\le\varepsilon,\quad\text{and}\ \|D\|_{\tilde f^s_{pq}\mapsto\tilde f^s_{pq}}\le\varepsilon,$$

for a sufficiently small ε, where the norm bounds and ε are uniform with respect to $(s,p,q)\in\Omega$. As in the classical case on $\mathbb{R}^n$ (see [7]), we shall show the boundedness of the above operators by using the machinery of almost diagonal operators.

It will be convenient to us to denote

$$\ell(\xi) := b^{-j}\quad\text{for}\quad \xi\in\mathcal{X}_j,\ j\ge 0. \tag{4.14}$$

Here $b\ge 2$ is the constant from the construction of the frames in §2.3. Evidently, $\ell(\xi)$ is a constant multiple of the radius of the neighborhood A_ξ of ξ.

Definition 4.3. *Let A be a linear operator acting on b^s_{pq}, $\tilde b^s_{pq}$, f^s_{pq}, or $\tilde f^s_{pq}$, with an associated matrix $(a_{\xi\eta})_{\xi,\eta\in\mathcal{X}}$. Let $\mathcal{J} := d/\min\{1,p\}$ in the case of the spaces b^s_{pq}, $\tilde b^s_{pq}$, and $\mathcal{J} := d/\min\{1,p,q\}$ for f^s_{pq}, $\tilde f^s_{pq}$. We say that A is almost diagonal if there exists $\delta>0$ such that*

$$\sup_{\xi,\eta\in\mathcal{X}}\frac{|a_{\xi\eta}|}{\omega_\delta(\xi,\eta)}<\infty,\quad\text{where}$$

$$\omega_\delta(\xi,\eta) := \left(\min\left\{\frac{\ell(\xi)}{\ell(\eta)},\frac{\ell(\eta)}{\ell(\xi)}\right\}\right)^{|s|+\mathcal{J}+\frac d2+\delta}\left(1+\frac{\rho(\xi,\eta)}{\max\{\ell(\xi),\ell(\eta)\}}\right)^{-|s|-\mathcal{J}-\frac d2-\delta}.$$

We next show that the almost diagonal operators are bounded on b^s_{pq}, $\tilde b^s_{pq}$, f^s_{pq}, and $\tilde f^s_{pq}$. More precisely, with the notation

$$\|A\|_\delta := \sup_{\xi,\eta\in\mathcal{X}}\frac{|a_{\xi\eta}|}{\omega_\delta(\xi,\eta)} \tag{4.15}$$

we have:

Theorem 4.4. *Suppose $s\in\mathbb{R}$ and $0<p,q<\infty$, and let $\|A\|_\delta<\infty$ (in the sense of Definition 4.3) for some $\delta>0$. Then there exists a constant $c>0$ such that for any sequence $h:=\{h_\xi\}_{\xi\in\mathcal{X}}\in b^s_{pq}$*

$$\|Ah\|_{b^s_{pq}}\le c\|A\|_\delta\|h\|_{b^s_{pq}}, \tag{4.16}$$

and the same holds true with b^s_{pq} replaced by $\tilde b^s_{pq}$, f^s_{pq}, or $\tilde f^s_{pq}$. Here the constant c can be written in the form $c=c_1(p+1)c_2^{|s|}c_3^{1/p+1/q}(1/q)^{1/q}$, where $c_1,c_2,c_3>1$ depend only on δ,b,γ, and c_0.

To streamline the presentation we divert the proof of this theorem to the appendix.

Remark. Observe that $\omega_\delta(\xi,\eta)$ in the definition of almost diagonal operators can be optimized depending on the specific space b^s_{pq}, $\tilde{b}^s_{pq}$, f^s_{pq}, or $\tilde{f}^s_{pq}$. As a result this would enable us to work with smaller parameters N and K in the construction of $\{\theta_\xi\}$ and in Theorem 4.2. However, we have no restrictions on N, K and opted to go for a simpler version of $\omega_\delta(\xi,\eta)$.

The above theorem and the construction from §3.2 indicate that to prove that $\{\theta_\xi\}$, $\{\tilde{\theta}_\xi\}$ is a pair of frames for B^s_{pq}, $\tilde{B}^s_{pq}$, F^s_{pq}, or $\tilde{F}^s_{pq}$ it suffices to show that the operators with matrices A, B, and D, defined in (4.13) are almost diagonal and $\|D\|_\delta \le \varepsilon$, for fixed $\delta>0$ and sufficiently small $\varepsilon>0$.

4.4. **Inner products.** We next estimate the inner product involved in (4.13) which in a sense characterize the localization and approximation properties of the new system $\{\theta_\xi\}$ relative to the old frame $\{\psi_\xi\}$.

Theorem 4.5. *For any $\xi\in\mathcal{X}_j$, $\eta\in\mathcal{X}_\ell$ we have*

$$|\langle\psi_\xi,\psi_\eta\rangle| \le cb^{-|j-\ell|(N-K-d)}\big(1+b^{\min\{j,\ell\}}\rho(\xi,\eta)\big)^{-K}, \tag{4.17}$$

$$|\langle\theta_\xi,\psi_\eta\rangle| \le cb^{-|j-\ell|(N-K-d)}\big(1+b^{\min\{j,\ell\}}\rho(\xi,\eta)\big)^{-K}, \tag{4.18}$$

and

$$|\langle\psi_\xi-\theta_\xi,\psi_\eta\rangle| \le c\varepsilon b^{-|j-\ell|(N-K-d)}\big(1+b^{\min\{j,\ell\}}\rho(\xi,\eta)\big)^{-K}, \tag{4.19}$$

where $c>0$ is a constant independent of ε. Moreover, the above inequalities hold with ψ_η replaced by $\tilde{\psi}_\eta$.

Proof. We shall only prove (4.19); the proof of (4.17) or (4.18) is similar and will be omitted. Assume $j,\ell\ge 1$. The other cases are similar. From (2.16) and (4.6) we get

$$\begin{aligned}
&|\langle\psi_\xi-\theta_\xi,\psi_\eta\rangle|\\
&\quad\le c|B(\xi,b^{-j})|^{1/2}|B(\eta,b^{-\ell})|^{1/2}\big\langle\big[\Psi(b^{-j}\sqrt{L})-\Theta(b^{-j}\sqrt{L})\big](\cdot,\xi),\Psi(2^{-\ell}\sqrt{L})(\cdot,\eta)\big\rangle\\
&\quad= c|B(\xi,b^{-j})|^{1/2}|B(\eta,b^{-\ell})|^{1/2}\big|\big[\Psi(b^{-j}\sqrt{L})-\Theta(b^{-j}\sqrt{L})\big]\Psi(2^{-\ell}\sqrt{L})(\xi,\eta)\big|
\end{aligned}$$

Two cases present themselves here.

Case 1: $\ell\ge j$. Set $F(\lambda):=[\Psi(\lambda)-\Theta(\lambda)]\Psi(b^{-(\ell-j)}\lambda)$. Evidently,

$$F(b^{-j}\sqrt{L}):=[\Psi(b^{-j}\sqrt{L})-\Theta(b^{-j}\sqrt{L})]\Psi(b^{-\ell}\sqrt{L}),\quad \operatorname{supp}F\subset[b^{\ell-j-1},b^{\ell-j+1}],$$

and by Proposition 4.1

$$\|F^{(\nu)}\|_\infty \le \frac{c\varepsilon}{b^{(\ell-j)N}},\quad \nu=0,1,\dots,K.$$

Now applying Theorem 2.2 we infer

$$\begin{aligned}
|F(b^{-j}\sqrt{L})(x,y)| &\le \frac{cb^{(\ell-j)d}\big(\|F\|_\infty+b^{(\ell-j)K}\|F^{(K)}\|_\infty\big)}{|B(x,b^{-j})|^{1/2}|B(y,b^{-j})|^{1/2}(1+b^j\rho(x,y))^K}\\
&\le \frac{c\varepsilon b^{-(\ell-j)(N-K-d)}}{|B(x,b^{-j})|^{1/2}|B(y,b^{-j})|^{1/2}(1+b^j\rho(x,y))^K}
\end{aligned}$$

and hence

$$|\langle\psi_\xi-\theta_\xi,\psi_\eta\rangle| \le \frac{c\varepsilon b^{-(\ell-j)(N-K-d)}}{(1+b^j\rho(\xi,\eta))^K},$$

which verifies (4.19).

Case 2: $\ell < j$. Set $F(\lambda) := [\Psi(b^{-(j-\ell)}\lambda) - \Theta(b^{-(j-\ell)}\lambda)]\Psi(\lambda)$. Evidently,

$$F(b^{-\ell}\sqrt{L}) := [\Psi(b^{-j}\sqrt{L}) - \Theta(b^{-j}\sqrt{L})]\Psi(b^{-\ell}\sqrt{L}), \quad \operatorname{supp} F \subset [b^{-1}, b^2],$$

and by Proposition 4.1

$$\|F^{(\nu)}\|_\infty \le c\varepsilon b^{-(j-\ell)N}, \quad \nu = 0, 1, \dots, K.$$

Now, again by Theorem 2.2

$$\begin{aligned} |F(b^{-\ell}\sqrt{L})(x,y)| &\le \frac{c\big(\|F\|_\infty + \|F^{(K)}\|_\infty\big)}{|B(x,b^{-\ell})|^{1/2}|B(y,b^{-\ell})|^{1/2}(1+b^\ell\rho(x,y))^K} \\ &\le \frac{c\varepsilon b^{-(j-\ell)N}}{|B(x,b^{-\ell})|^{1/2}|B(y,b^{-\ell})|^{1/2}(1+b^\ell\rho(x,y))^K} \end{aligned}$$

and hence

$$|\langle \psi_\xi - \theta_\xi, \psi_\eta\rangle| \le \frac{c\varepsilon b^{-(j-\ell)N}}{(1+b^\ell\rho(\xi,\eta))^K},$$

which confirms (4.19). $\square$

4.5. **Proof of Theorem 4.2.** Observe first that a careful examination of the development in [12] shows that the pair of frames $\{\psi_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde\psi_\xi\}_{\xi\in\mathcal{X}}$, constructed in [12], satisfy condition **A1-A2** in §3.1 with $\mathcal{B}$, $\mathcal{B}_d$ being any of the pairs of spaces B^s_{pq}, b^s_{pq} or $\tilde B^s_{pq}$, $\tilde b^s_{pq}$ or F^s_{pq}, f^s_{pq} or $\tilde F^s_{pq}$, $\tilde f^s_{pq}$, and all relevant constants, in particular, the constants in Theorems 2.10-2.11, are uniform with respect to $(s,p,q)\in\Omega$, where Ω is defined in (4.8). In fact, the maximal inequality (2.40) is the main nontrivial contributor to the constants of interest in [12]. Condition **A3** (§3.1) is also satisfied since we assume $p, q < \infty$. The validity of condition **A4** is included in the argument in what follows.

Note that, if $(s,p,q)\in\Omega$, then the constant c from Theorem 4.4 applied with e.g. $\delta = 1$ can be bounded as follows

$$c \le c_1(p_1+1)c_2^{s_0}c_3^{1/p_0+1/q_0}(1/q_0)^{1/q_0},$$

where the constants $c_1, c_2, c_3 > 0$ depend only on b, γ, c_0. Here, any $\delta > 0$ would do the job. Therefore, Theorem 4.2 will follow from Theorem 3.5 if we prove that the operators with matrices A, B, D defined in (4.13) are almost diagonal with $\delta = 1$ on b^s_{pq}, $\tilde b^s_{pq}$, f^s_{pq}, or $\tilde f^s_{pq}$ and in addition for sufficiently small $\varepsilon > 0$

$$\|D\|_\delta \le \varepsilon \quad \text{with } \delta = 1. \tag{4.20}$$

We shall only prove (4.20); the boundedness of the operators associated with the other matrices follows similarly. Recall that

$$D := (d_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad d_{\xi,\eta} := \langle \psi_\eta - \theta_\eta, \psi_\xi\rangle.$$

It will be convenient to introduce the more detailed notation $\omega_\delta(\xi,\eta; s, \mathcal{J})$ for the quantity $\omega_\delta(\xi,\eta)$ from Definition 4.3. We claim that using that $K \ge s_0 + \mathcal{J}_0 + d/2 + 1$ and $N \ge K + s_0 + \mathcal{J}_0 + 3d/2 + 1$ it follows that

$$|d_{\xi,\eta}| := |\langle \psi_\eta - \theta_\eta, \psi_\xi\rangle| \le c\varepsilon\omega_1(\xi,\eta; s_0, \mathcal{J}_0), \quad \xi,\eta\in\mathcal{X}, \tag{4.21}$$

where the constant c is independent of ε. This along with the obvious fact that $\omega_1(\xi,\eta; s, \mathcal{J}) \ge \omega_1(\xi,\eta; s_0, \mathcal{J}_0)$, whenever $(s,p,q)\in\Omega$, yields

$$\|D\|_1 := \sup_{\xi,\eta\in\mathcal{X}} \frac{|d_{\xi,\eta}|}{\omega_1(\xi,\eta; s,\mathcal{J})} \le c\varepsilon.$$

However, ε is independent of M, N, and c. Therefore, $c\varepsilon$ above can be replaced by ε and (4.20) would hold.

For the proof of (4.21) consider the case when $\ell(\xi) \ge \ell(\eta)$, i.e. $\xi \in \mathcal{X}_j, \eta \in \mathcal{X}_\ell$ and $\ell \ge j$. From estimate (4.19) in Theorem 4.5 we get

$$\begin{aligned}|a_{\xi,\eta}| &\le c\varepsilon b^{-|j-\ell|(N-K-d)}(1+2^j\rho(\xi,\eta))^{-K}\\ &= c\varepsilon\Big(\frac{\ell(\eta)}{\ell(\xi)}\Big)^{N-K-d}\Big(1+\frac{\rho(\xi,\eta)}{\ell(\xi)}\Big)^{-K} \le c\varepsilon\omega_1(\xi,\eta;s_0,\mathcal{J}_0),\end{aligned}$$

where in the last inequality we used that $K \ge s_0 + \mathcal{J}_0 + d/2 + 1$ and $N \ge K + s_0 + \mathcal{J}_0 + 3d/2 + 1$.

The proof of (4.21) in the case $\ell(\xi) < \ell(\eta)$ is the same and will be omitted.

The claimed properties of the dual frame elements $\tilde{\theta}_\xi$, $\xi \in \mathcal{X}$, are established in Theorem 4.6 below. $\square$

4.6. **Localization of $\tilde{\theta}_\xi$.** From our general construction of new frames in §3 it only follows that the dual frame elements $\tilde{\theta}_\xi$, $\xi \in \mathcal{X}$, are continuous linear functional on the underlying space $\mathcal{B}$, that is, the respective B- or F-space in the current setting. Now, we would like to provide more information about the dual frame elements, and in particular, to identify them with well localized functions.

Theorem 4.6. *For any $\gamma, \sigma > 0$ the parameters K, N and ε in the construction of $\{\tilde{\theta}_\xi\}$ can be selected so that for any $\xi \in \mathcal{X}_j$, $j \ge 0$, the linear functional $\tilde{\theta}_\xi$ can be identified with a function*

$$\tilde{\theta}_\xi = \sum_{\nu\ge 0}\sum_{\eta\in\mathcal{X}_\nu}\alpha_{\xi\eta}\tilde{\psi}_\eta, \quad \textit{where} \quad |\alpha_{\xi\eta}| \le \frac{cb^{-|j-\nu|\gamma}}{\big(1+b^{j\vee\nu}\rho(\xi,\eta)\big)^\sigma}, \tag{4.22}$$

and

$$|\tilde{\theta}_\xi(x)| \le \frac{c|B(\xi,b^{-j})|^{-1/2}}{\big(1+b^j\rho(x,\xi)\big)^\sigma}, \quad x \in M. \tag{4.23}$$

The following two lemmas will be instrumental in the proof of this theorem.

Lemma 4.7. *Let $\sigma \ge 2d+1$, $b > 1$, $0 \le s,t \le m$, and $x, y \in M$. Then*

$$\sum_{\omega\in\mathcal{X}_m}\frac{1}{\big(1+b^s\rho(x,\omega)\big)^\sigma\big(1+b^t\rho(y,\omega)\big)^\sigma} \le \frac{cb^{(m-s\vee t)\sigma}}{\big(1+b^{s\wedge t}\rho(x,y)\big)^\sigma}, \tag{4.24}$$

where $c > 0$ depends only on d, b and σ.

Proof. Assume $0 \le s \le t \le m$. Denote the quantity on the left in (4.24) by Σ and set

$$\mathcal{X}_m^1 := \{\omega \in \mathcal{X}_m : \rho(x,\omega) \ge \rho(x,y)/2\}, \quad \mathcal{X}_m^2 := \{\omega \in \mathcal{X}_m : \rho(y,\omega) > \rho(x,y)/2\}.$$

Then Σ can be represented as $\Sigma \le \sum_{\omega\in\mathcal{X}_m^1}\cdots + \sum_{\omega\in\mathcal{X}_m^2}\cdots =: \Sigma_1 + \Sigma_2$. For the first sum we have

$$\Sigma_1 \le \frac{cb^{(m-t)\sigma}}{\big(1+b^s\rho(x,y)\big)^\sigma}\sum_{\omega\in\mathcal{X}_m}\frac{1}{\big(1+b^m\rho(y,\omega)\big)^\sigma} \le \frac{cb^{(m-t)\sigma}}{\big(1+b^s\rho(x,y)\big)^\sigma}.$$

Here we used the following simple inequality

$$\sum_{\omega\in\mathcal{X}_m}\big(1+b^m\rho(y,\omega)\big)^{-2d-1} \le c < \infty, \tag{4.25}$$

see [2], inequality (2.20).

To estimate Σ_2 we consider two cases depending on whether $b^s\rho(x,y) \ge 1$ or $b^s\rho(x,y) < 1$. In the first case, just as above we get

$$\Sigma_2 \le \frac{cb^{(m-s)\sigma}}{\big(1+b^t\rho(x,y)\big)^\sigma} \le \frac{cb^{(m-s)\sigma}}{\big(b^t\rho(x,y)\big)^\sigma} = \frac{cb^{(m-t)\sigma}}{\big(b^s\rho(x,y)\big)^\sigma} \le \frac{c2^\sigma b^{(m-t)\sigma}}{\big(1+b^s\rho(x,y)\big)^\sigma}.$$

If $b^s\rho(x,y) < 1$, then using (4.25) we obtain

$$\Sigma_2 \le \sum_{\omega\in\mathcal{X}_m} \frac{1}{\big(1+b^t\rho(y,\omega)\big)^\sigma} \le \sum_{\omega\in\mathcal{X}_m} \frac{b^{(m-t)\sigma}}{\big(1+b^m\rho(y,\omega)\big)^\sigma} \le cb^{(m-t)\sigma} \le \frac{c2^\sigma b^{(m-t)\sigma}}{\big(1+b^s\rho(x,y)\big)^\sigma}.$$

The above estimates for Σ_1 and Σ_2 yield (4.24). $\square$

Lemma 4.8. *Let $\sigma \ge 2d+1$ and $j,\nu \ge 0$, $\delta > 0$, $b > 1$, and $x, y \in M$. Then*

$$\sum_{m\ge 0}\sum_{\omega\in\mathcal{X}_m} \frac{b^{-|m-j|\sigma}}{\big(1+b^{j\wedge m}\rho(x,\omega)\big)^\sigma} \frac{b^{-|m-\nu|(\sigma+\delta)}}{\big(1+b^{\nu\wedge m}\rho(y,\omega)\big)^\sigma} \le \frac{c_\diamond b^{|j-\nu|\sigma}}{\big(1+b^{j\wedge\nu}\rho(x,y)\big)^\sigma} \tag{4.26}$$

and

$$\sum_{m\ge 0}\sum_{\omega\in\mathcal{X}_m} \frac{b^{-|m-j|(\sigma+\delta)}}{\big(1+b^{j\wedge m}\rho(x,\omega)\big)^\sigma} \frac{1}{\big(1+b^{m}\rho(y,\omega)\big)^\sigma} \le \frac{c_\diamond}{\big(1+b^{j}\rho(x,y)\big)^\sigma}, \tag{4.27}$$

where $c_\diamond > 0$ depends only on $d, b, \delta,$ and σ.

Proof. Assume $\nu \le j$ and denote by Σ the quantity on the left in (4.26). We split Σ into three $\Sigma = \sum_{0\le m<\nu}\cdots + \sum_{\nu\le m\le j}\cdots + \sum_{m>j}\cdots =: \Sigma_1 + \Sigma_2 + \Sigma_3$. Now, using Lemma 4.7

$$\begin{aligned}
\Sigma_1 &= \sum_{0\le m<\nu}\sum_{\omega\in\mathcal{X}_m} \frac{b^{-(j-m)\sigma}}{\big(1+b^{m}\rho(x,\omega)\big)^\sigma} \frac{b^{-(\nu-m)(\sigma+\delta)}}{\big(1+b^{m}\rho(y,\omega)\big)^\sigma}\\
&\le \sum_{0\le m<\nu} \frac{cb^{-(j-m)\sigma}b^{-(\nu-m)(\sigma+\delta)}}{\big(1+b^{m}\rho(x,y)\big)^\sigma}\\
&\le \frac{c}{\big(1+b^{\nu}\rho(x,y)\big)^\sigma}\sum_{0\le m<\nu} b^{-(j-m)\sigma} \le \frac{cb^{-(j-\nu)\sigma}}{\big(1+b^{\nu}\rho(x,y)\big)^\sigma}.
\end{aligned}$$

We estimate Σ_2 using again (4.24)

$$\begin{aligned}
\Sigma_2 &= \sum_{\nu\le m\le j}\sum_{\omega\in\mathcal{X}_m} \frac{b^{-(j-m)\sigma}}{\big(1+b^{m}\rho(x,\omega)\big)^\sigma} \frac{b^{-(m-\nu)(\sigma+\delta)}}{\big(1+b^{\nu}\rho(y,\omega)\big)^\sigma}\\
&\le \sum_{\nu\le m\le j} \frac{cb^{-(j-m)\sigma}b^{-(m-\nu)(\sigma+\delta)}}{\big(1+b^{\nu}\rho(x,y)\big)^\sigma}\\
&= \frac{cb^{-(j-\nu)\sigma}}{\big(1+b^{\nu}\rho(x,y)\big)^\sigma}\sum_{\nu\le m\le j} b^{-(m-\nu)\delta} \le \frac{cb^{-(j-\nu)\sigma}}{\big(1+b^{\nu}\rho(x,y)\big)^\sigma}.
\end{aligned}$$

To estimate Σ_3 we proceed in the same way

$$\Sigma_3 = \sum_{m>j}\sum_{\omega\in\mathcal{X}_m} \frac{b^{-(m-j)\sigma}}{\big(1+b^j\rho(x,\omega)\big)^\sigma}\frac{b^{-(m-\nu)(\sigma+\delta)}}{\big(1+b^\nu\rho(y,\omega)\big)^\sigma}$$
$$\le \sum_{m>j}\frac{cb^{-(m-\nu)(\sigma+\delta)}}{\big(1+b^\nu\rho(x,y)\big)^\sigma} \le \frac{cb^{-(j-\nu)\sigma}}{\big(1+b^\nu\rho(x,y)\big)^\sigma}.$$

The above estimates for Σ_1, Σ_2, Σ_3 yield (4.26). The proof of (4.26) when $\nu > j$ follows the same lines. The proof of (4.27) is similar and simpler; we omit it. $\square$

Proof of Theorem 4.6. Clearly, it suffices to prove the theorem only whenever $\gamma = \sigma \ge 5d/2+2$. Given $\sigma \ge 5d/2+2$ we impose on the parameters K, M from Theorem 4.2 the additional conditions: $N-K-d \ge \sigma+1$ and $K \ge \sigma$. Later on an additional condition will be imposed on ε as well. By Theorem 4.5 we have for $\xi \in \mathcal{X}_j$ and $\eta \in \mathcal{X}_\nu$, $j,\nu \ge 0$,

$$|\langle \psi_\xi - \theta_\xi, \tilde\psi_\eta\rangle| \le c_\flat \varepsilon b^{-|j-\nu|(\sigma+1)}\big(1+b^{j\wedge\nu}\rho(\xi,\eta)\big)^{-\sigma} \tag{4.28}$$

and

$$|\langle \psi_\xi, \tilde\psi_\eta\rangle| \le cb^{-|j-\nu|\sigma}\big(1+b^{j\wedge\nu}\rho(\xi,\eta)\big)^{-\sigma}. \tag{4.29}$$

Note that by (4.11) the linear functional $\tilde\theta_\xi$ can be identified with

$$\tilde\theta_\xi = \sum_{\eta\in\mathcal{X}} \overline{\langle T^{-1}\psi_\eta, \tilde\psi_\xi\rangle}\tilde\psi_\eta, \tag{4.30}$$

and our next step is to obtain a suitable representation for $T^{-1}\psi_\eta$.

Lemma 4.9. *For any $\sigma > 0$ the parameters K, N, and ε in the construction of $\{\tilde\theta_\xi\}$ can be selected so that for any $\eta \in \mathcal{X}_\nu$, $\nu \ge 0$, we have*

$$T^{-1}\psi_\eta = \sum_{m\ge 0}\sum_{\omega\in\mathcal{X}_m} t_{\eta\omega}(\psi_\omega - \theta_\omega), \quad \textit{where} \quad |t_{\eta\omega}| \le \frac{cb^{-|\nu-m|\sigma}}{\big(1+b^{\nu\wedge m}\rho(\eta,\omega)\big)^\sigma}, \tag{4.31}$$

and

$$|T^{-1}\psi_\eta(x)| \le \frac{c|B(\eta,b^{-\nu})|^{-1/2}}{\big(1+b^\nu\rho(\eta,x)\big)^\sigma}, \quad x \in M. \tag{4.32}$$

The above series converges uniformly on M.

Proof. From the construction of $\{\tilde\theta_\xi\}$ in §3.2 (Lemma 3.2) we have

$$T^{-1}f = \sum_{k\ge 1}(I-T)^k f, \quad \text{where} \quad Tf := \sum_{\xi\in\mathcal{X}}\langle f, \tilde\psi_\xi\rangle\theta_\xi,$$

for any distribution f from the underlying B- or F-space $\mathcal{B}$ with convergence in the norm of the space and as a consequence in $\mathcal{D}'$. From this and the representation $f = \sum_{\omega\in\mathcal{X}}\langle f, \tilde\psi_\omega\rangle\psi_\omega$ (Proposition 2.5) we infer

$$(I-T)f = \sum_{m\ge 0}\sum_{\omega\in\mathcal{X}_m}\langle f, \tilde\psi_\omega\rangle(\psi_\omega - \theta_\omega). \tag{4.33}$$

We apply the above to ψ_η ($\eta \in \mathcal{X}_\nu$, $\nu \ge 0$). We claim that for any $k \ge 1$ we have

$$(I-T)^k\psi_\eta = \sum_{m\ge 0}\sum_{\omega\in\mathcal{X}_m} T^k_{\eta\omega}(\psi_\omega - \theta_\omega), \tag{4.34}$$

where the convergence is in $\mathcal{B}$ and hence in $\mathcal{D}'$, and

$$|T^k_{\eta\omega}| \le \frac{c(c_*\varepsilon)^{k-1}b^{-|\nu-m|\sigma}}{\big(1+b^{\nu\wedge m}\rho(\eta,\omega)\big)^\sigma}, \quad \omega\in\mathcal{X}_m, \quad c_* := c_\diamond c_\flat. \tag{4.35}$$

Here the constants $c_\flat$, $c_\diamond$ are from Lemma 4.8 and (4.28).

Indeed, using (4.33) identity (4.34) holds for $k=1$ with $T^1_{\eta\omega} = \langle\psi_\eta, \tilde\psi_\omega\rangle$ and by (4.29) it follows that inequality (4.35) holds for $k=1$. Assume now that (4.34)-(4.35) hold for some $k\ge 1$. Then

$$(I-T)^{k+1}\psi_\eta = \sum_{m\ge 0}\sum_{\omega\in\mathcal{X}_m}\langle (I-T)^k\psi_\eta, \tilde\psi_\omega\rangle(\psi_\omega-\theta_\omega)$$

and using (4.28), (4.35), and Lemma 4.8 we obtain ($\eta\in\mathcal{X}_\nu$, $\omega\in\mathcal{X}_m$)

$$\begin{aligned}|\langle (I-T)^k\psi_\eta, \tilde\psi_\omega\rangle| &\le \sum_{\ell\ge 0}\sum_{\alpha\in\mathcal{X}_\ell}|T^k_{\eta\alpha}||\langle\psi_\alpha-\theta_\alpha, \tilde\psi_\omega\rangle| \\ &\le c(c_*\varepsilon)^{k-1}c_\flat\varepsilon\sum_{\ell\ge 0}\sum_{\alpha\in\mathcal{X}_\ell}\frac{b^{-|\nu-\ell|\sigma}}{\big(1+b^{\nu\wedge\ell}\rho(\eta,\alpha)\big)^\sigma}\frac{b^{-|m-\ell|(\sigma+1)}}{\big(1+b^{m\wedge\ell}\rho(\omega,\alpha)\big)^\sigma} \\ &\le c(c_*\varepsilon)^k\frac{b^{-|\nu-m|\sigma}}{\big(1+b^{\nu\wedge m}\rho(\eta,\omega)\big)^\sigma}, \quad c_* := c_\diamond c_\flat.\end{aligned}$$

Therefore, by induction (4.34)-(4.35) hold for all $k\ge 1$.

We now impose on ε the additional condition $\varepsilon \le \frac{1}{2c_*} = \frac{1}{2c_\diamond c_\flat}$. Summing up we obtain

$$\sum_{k\ge 1}|T^k_{\eta\omega}| \le \frac{cb^{-|\nu-m|\sigma}}{\big(1+b^{\nu\wedge m}\rho(\eta,\omega)\big)^\sigma}\sum_{k\ge 1}(c_*\varepsilon)^{k-1} \le \frac{2cb^{-|\nu-m|\sigma}}{\big(1+b^{\nu\wedge m}\rho(\eta,\omega)\big)^\sigma}.$$

This, the representation $T^{-1}\psi_\eta = \sum_{k\ge 1}(I-T)^k\psi_\eta$, and (4.34)-(4.35) imply (4.31).

By the localization of ψ_ξ and $\tilde\psi_\xi$, given in Proposition 2.5, it follows that

$$|\psi_\xi(x)|, |\tilde\psi_\xi(x)| \le \frac{c|B(\xi,b^{-j})|^{-1/2}}{\big(1+b^j\rho(x,\xi)\big)^\sigma}, \quad x\in M,\ \xi\in\mathcal{X}_j,\ j\ge 0. \tag{4.36}$$

On the other hand, by (4.6)-(4.7) it follows that

$$\|\theta_\xi\|_\infty \le c|B(\xi,b^{-j})|^{-1/2} \quad\text{and}\quad \operatorname{supp}\theta_\xi\subset B(\xi,cb^{-j}) \quad\text{for}\ \ \xi\in\mathcal{X}_j.$$

Therefore,

$$|\psi_\xi(x)-\theta_\xi(x)| \le \frac{c|B(\xi,b^{-j})|^{-1/2}}{\big(1+b^j\rho(x,\xi)\big)^\sigma}, \quad x\in M,\ \xi\in\mathcal{X}_j.$$

This along with the estimate for $|t_{\eta\omega}|$ in (4.31) yield

$$\begin{aligned}|T^{-1}\psi_\eta(x)| &\le \sum_{m\ge 0}\sum_{\omega\in\mathcal{X}_m}|t_{\eta\omega}||\psi_\omega(x)-\theta_\omega(x)| \\ &\le c\sum_{m\ge 0}\sum_{\omega\in\mathcal{X}_m}\frac{b^{-|\nu-m|\sigma}}{\big(1+b^{\nu\wedge m}\rho(\eta,\omega)\big)^\sigma}\frac{|B(\omega,b^{-m})|^{-1/2}}{\big(1+b^m\rho(x,\omega)\big)^\sigma}.\end{aligned}$$

By (1.2) and (2.2) it readily follows that

$$|B(\eta,b^{-\nu})| \le c_0^2 b^{|\nu-m|d}\big(1+b^{\nu\wedge m}\rho(\eta,\omega)\big)^d|B(\omega,b^{-m})|.$$

We insert this above and obtain

$$
\begin{aligned}
|T^{-1}\psi_\eta(x)| &\le c|B(\eta,b^{-\nu})|^{-1/2}\sum_{m\ge 0}\sum_{\omega\in\mathcal{X}_m}\frac{b^{-|\nu-m|(\sigma-d/2)}}{\big(1+b^{\nu\wedge m}\rho(\eta,\omega)\big)^{\sigma-d/2}}\frac{1}{\big(1+b^m\rho(x,\omega)\big)^{\sigma}}\\
&\le \frac{c|B(\eta,b^{-\nu})|^{-1/2}}{\big(1+b^{\nu}\rho(\eta,x)\big)^{\sigma-d/2-1}}.
\end{aligned}
$$

Here for the last inequality we used (4.27) with σ replaced by $\sigma-d/2-1\ge 2d+1$. Finally, observe that since σ can be selected selected arbitrarily large then above σ can be replaced by $\sigma+d/2+1$, which leads to (4.32). This completes the proof of the lemma. □

We are now ready to complete the proof of Theorem 4.6. Using Lemmas 4.8-4.9 we obtain

$$
\begin{aligned}
|\langle T^{-1}\psi_\eta,\tilde{\psi}_\xi\rangle| &\le \sum_{m\ge 0}\sum_{\omega\in\mathcal{X}_m}|t_{\eta\omega}||\langle\psi_\omega-\theta_\omega,\tilde{\psi}_\xi\rangle|\\
&\le c\sum_{m\ge 0}\sum_{\omega\in\mathcal{X}_m}\frac{b^{-|\nu-m|\sigma}}{\big(1+b^{\nu\wedge m}\rho(\eta,\alpha)\big)^{\sigma}}\frac{b^{-|m-j|(\sigma+1)}}{\big(1+b^{m\wedge j}\rho(\omega,\alpha)\big)^{\sigma}}\\
&\le \frac{cb^{-|\nu-j|\sigma}}{\big(1+b^{\nu\wedge j}\rho(\eta,\xi)\big)^{\sigma}}.
\end{aligned}
$$

Using this in (4.30) implies (4.22) with $\gamma=\sigma$.

To establish (4.23) we use the estimate for $|\alpha_{\xi\eta}|$ in (4.22) (with $\gamma=\sigma$) and the localization of $\tilde{\psi}_\xi$ from (4.36). We get

$$
\begin{aligned}
|\tilde{\theta}_\xi(x)| &\le \sum_{\nu\ge 0}\sum_{\eta\in\mathcal{X}_\nu}|\langle T^{-1}\psi_\eta,\tilde{\psi}_\xi\rangle||\tilde{\psi}_\eta(x)|\\
&\le c\sum_{\nu\ge 0}\sum_{\eta\in\mathcal{X}_\nu}\frac{b^{-|j-\nu|\sigma}}{\big(1+b^{j\vee\nu}\rho(\xi,\eta)\big)^{\sigma}}\frac{|B(\eta,b^{-\nu})|^{-1/2}}{\big(1+b^{\nu}\rho(x,\eta)\big)^{\sigma}}.
\end{aligned}
$$

Now, just as in the proof of Lemma 4.9 we conclude that (4.23) holds true. □

5. Application of compactly supported frames to Hardy spaces

In this section we consider atomic Hardy spaces H^p_A, $0<p\le 1$, in the general setting of this article (§1). We use the compactly supported frames from §4 to establish Littlewood-Paley characterization, and as a consequence, frame decomposition of the atomic Hardy spaces H^p_A. This result can also be viewed as an atomic decomposition of the Triebel-Lizorkin spaces F^0_{p2}, $0<p\le 1$.

Inhomogeneous atomic Hardy spaces. In introducing atoms we follow to a large extent [10, 4]. The inhomogeneous nature of our setting, however, compels us to introduce two kinds of atoms.

Definition 5.1. *Let $0<p\le 1$ and $n:=\lfloor d/2p\rfloor+1$, where d is from* (1.2)*. A function a is called an atom (of type A or B) associated with the operator L if it satisfies one of the following sets of conditions:*

(*A*) *There exists a ball B of radius $r=r_B$, $r\ge 1$, such that*

(i) $\operatorname{supp} a\subset B$ *and*

(ii) $\|a\|_{L^2}\le|B|^{1/2-1/p}$.

(B) There exists a function $b \in D(L^n)$ and a ball B of radius $r = r_B$, $r > 0$ such that

(i) $a = L^n b$,
(ii) $\operatorname{supp} L^k b \subset B$, $k = 0, 1, \dots, n$, *and*
(iii) $\|L^k b\|_{L^2} \le r^{2(n-k)} |B|^{1/2-1/p}$, $k = 0, 1, \dots, n$.

Being in a setting different from the one in [10, 4] we define the atomic Hardy spaces H^p_A as spaces of distributions (§2.4).

Definition 5.2. *The atomic Hardy space H^p_A, $0 < p \le 1$, is defined as the set of all distributions $f \in \mathcal{D}'$ that can be represented in the form*

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \text{where} \quad \sum_{j=1}^{\infty} |\lambda_j|^p < \infty, \tag{5.1}$$

$\{a_j\}$ are atoms, and the convergence is in $\mathcal{D}'$. We set

$$\|f\|_{H^p_A} := \inf_{f = \sum_{j \ge 1} \lambda_j a_j} \Big(\sum_{j \ge 1} |\lambda_j|^p \Big)^{1/p}, \quad f \in H^p_A. \tag{5.2}$$

Our first order of business is to give an example of atoms.

Lemma 5.3. *Assume that the constant N from the construction of Θ in Proposition 4.1 obeys the condition $N \ge 2n = 2\lfloor d/2p \rfloor + 2$.*

(i) For any $\xi \in \mathcal{X}_0$ the function

$$a_\xi := |B(\xi, 1)|^{1/2-1/p} \theta_\xi \quad \text{with} \quad \operatorname{supp} a_\xi \subset B(\xi, \tilde{c}R),$$

is a constant multiple of an atom of type A.

(ii) For any $\xi \in \mathcal{X}_j$, $j \ge 1$, the function

$$a_\xi := |B(\xi, b^{-j})|^{1/2-1/p} \theta_\xi \quad \text{with} \quad \operatorname{supp} a_\xi \subset B(\xi, \tilde{c}Rb^{-j}),$$

is a constant multiple of an atom of type B.

Above the constants $\tilde{c}, R$ are from (4.7).

Proof. Part (i) is immediate from the construction of θ_ξ, $\xi \in \mathcal{X}_0$.

To prove Part (ii) we put

$$b_\xi(x) := |B(\xi, b^{-j})|^{1/2-1/p} |A_\xi|^{1/2} L^{-n} \Theta(b^{-j}\sqrt{L})(x, \xi) \quad \text{for } \xi \in \mathcal{X}_j,\ j \ge 1.$$

Clearly, $L^n b_\xi = a_\xi$ and

$$\begin{aligned} L^k b_\xi(x) &= |B(\xi, b^{-j})|^{1/2-1/p} |A_\xi|^{1/2} L^{-(n-k)} \Theta(b^{-j}\sqrt{L})(x, \xi) \\ &= |B(\xi, b^{-j})|^{1/2-1/p} |A_\xi|^{1/2} b^{-2j(n-k)} g(b^{-j}\sqrt{L})(x, \xi), \end{aligned} \tag{5.3}$$

where $g(t) := t^{-2(n-k)} \Theta(t)$. By Proposition 4.1 $\operatorname{supp} \hat{g} \subset [-R, R]$ and applying Proposition 2.1 we obtain $\operatorname{supp} L^k b_\xi = g(b^{-j}\sqrt{L})(\cdot, \xi) \subset B(\xi, r)$ with $r = \tilde{c}Rb^{-j}$, $k = 0, 1, \dots, n$.

On the other hand, by Theorem 2.2 it follows that

$$\|g(b^{-j}\sqrt{L})(\cdot, \xi)\|_\infty \le c|B(\xi, b^{-j})|^{-1}$$

and we know that $|A_\xi| \le |B(\xi, b^{-j})|$, $\xi \in \mathcal{X}_j$. These coupled with (5.3) imply

$$\|L^k b_\xi\|_\infty \le c b^{-2j(n-k)} |B(\xi, b^{-j})|^{-1/p} \le c' r^{2(n-k)} |B(\xi, r)|^{-1/p}, \tag{5.4}$$

where the constant $c' > 0$ depends on $b, R, \tilde{c}, n$. Here for the last inequality we used (1.2). Now, the estimate $\|L^k b_\xi\|_{L^2} \le cr^{2(n-k)}|B(\xi,r)|^{1/2-1/p}$ follows by (5.4) and $\operatorname{supp} L^k b_\xi \subset B(\xi, r)$. $\square$

We now come to the main result in this section.

Theorem 5.4. *We have* $H^p_A = F^0_{p2}$, $0 < p \le 1$, *and*

$$\|f\|_{H^p_A} \sim \|f\|_{F^0_{p2}} \quad \text{for} \quad f \in H^p_A. \tag{5.5}$$

Proof. For the proof of the estimate $\|f\|_{F^0_{p2}} \le c\|f\|_{H^p_A}$, $f \in H^p_A$, we need this lemma:

Lemma 5.5. *For any atom* a *and* $0 < p \le 1$, *we have*

$$\|a\|_{F^0_{p2}} \le c < \infty. \tag{5.6}$$

Proof. Let a be an atom of type B in the sense of Definition 5.1 and suppose $\operatorname{supp} a \subset B$, $B = B(z,r)$. Denote briefly $B_2 := B(z, 2r)$. Let $\{\varphi_j\}_{j\ge0}$ be the functions from the definition of the B- and F-spaces in §2.4. From Spectral theory it follows that $Tf := \Big(\sum_{j\ge0} |\varphi_j(\sqrt{L})f(\cdot)|^2\Big)^{1/2}$ is a bounded operator on $L^2(M)$. Therefore,

$$\begin{aligned}
\Big\|\Big(\sum_{j\ge0} |\varphi_j(\sqrt{L})a(\cdot)|^2\Big)^{1/2}\Big\|_{L^p(B_2)} &\le \Big\|\Big(\sum_{j\ge0} |\varphi_j(\sqrt{L})a(\cdot)|^2\Big)^{1/2}\Big\|_{L^2(B_2)} |B_2|^{1/p-1/2} \\
&\le \Big\|\Big(\sum_{j\ge0} |\varphi_j(\sqrt{L})a(\cdot)|^2\Big)^{1/2}\Big\|_{L^2(M)} |B_2|^{1/p-1/2} \\
&\le c\|a\|_{L^2}|B|^{1/p-1/2} \le c,
\end{aligned}$$

where we used Hölder's inequality and that $\|a\|_{L^2} \le |B|^{1/2-1/p}$.

To estimate $\Big\|\Big(\sum_{j\ge0} |\varphi_j(\sqrt{L})a(\cdot)|^2\Big)^{1/2}\Big\|_{L^p(M\setminus B_2)}$ we split the index set into two, depending on whether $2^j \ge 1/r$ or $2^j < 1/r$.

Let $2^j \ge 1/r$. From Theorem 2.2 and (2.3) it follows that for any $\sigma > 0$ and $j \ge 1$

$$|\varphi_j(\sqrt{L})(x,y)| = |\varphi(2^{-j}\sqrt{L})(x,y))| \le c_\sigma |B(y, 2^{-j})|^{-1}\big(1 + 2^j\rho(x,y)\big)^{-\sigma}. \tag{5.7}$$

For the same reason this estimate holds for $j = 0$ as well. We choose $\sigma > d(2+1/p)$.

Let $x \in M \setminus B_2$ and $y \in B$. By (1.2) and using that $\rho(x,z) \ge r$ and $r2^j > 1$ we get

$$|B| = |B(z,r)| \le c_0(r2^j)^d |B(z, 2^{-j})| \le c_0\big(1 + 2^j\rho(x,z)\big)^d |B(z, 2^{-j})|.$$

On the other hand, by (2.2) and since $\rho(z,y) \le r \le \rho(x,z)$ we have

$$|B(z,2^{-j})| \le c_0\big(1 + 2^j\rho(z,y)\big)^d |B(y, 2^{-j})| \le c_0\big(1 + 2^j\rho(x,z)\big)^d |B(y,2^{-j})|$$

Therefore,

$$|B| \le c_0^2\big(1 + 2^j\rho(x,z)\big)^{2d}|B(y,2^{-j})|.$$

We use this and the obvious inequalities $\rho(x,z) \le \rho(x,y) + \rho(y,z) \le 2\rho(x,y)$ in (5.7) to obtain

$$|\varphi_j(\sqrt{L})(x,y)| \le c|B|^{-1}\big(1 + 2^j\rho(x,z)\big)^{-\sigma+2d}, \quad x \in M \setminus B_2,\ y \in B.$$

In turn, this and the fact that $\operatorname{supp} a \subset B$ and $\|a\|_2 \le |B|^{1/2-1/p}$ lead to

$$|\varphi_j(\sqrt{L})a(x)| = \Big|\int_B \varphi(2^{-j}\sqrt{L})(x,y)a(y)d\mu(y)\Big| \le \|a\|_{L^2}\|\varphi(2^{-j}\sqrt{L})(x,\cdot)\|_{L^2(B)}$$
$$\le |B|^{1-1/p}\|\varphi(2^{-j}\sqrt{L})(x,\cdot)\|_{L^\infty(B)} \le \frac{c|B|^{-1/p}}{(1+2^j\rho(x,z))^{\sigma_1}}$$

for $x \in M \setminus B_2$ with $\sigma_1 := \sigma - 2d > 0$. Summing up using that $\rho(x,z) \ge r \ge 2^{-j}$ we infer

$$\sum_{2^j \ge 1/r} |\varphi_j(\sqrt{L})a(x)|^2 \le c|B|^{-2/p} \sum_{2^j\ge 1/r} \frac{1}{(1+2^j\rho(x,z))^{2\sigma_1}} \le \frac{c|B|^{-2/p}}{(1+r^{-1}\rho(x,z))^{2\sigma_1}}.$$

Therefore,

$$\Big\|\Big(\sum_{2^j\ge 1/r} |\varphi_j(\sqrt{L})a(\cdot)|^2\Big)^{1/2}\Big\|^p_{L^p(M\setminus B_2)} \le c\int_M \frac{|B|^{-1}d\mu(x)}{(1+r^{-1}\rho(x,z))^{p\sigma_1}} \le c. \tag{5.8}$$

For the last inequality we used (2.4) and that $p\sigma_1 = p(\sigma-2d) > d$.

Let $2^j < 1/r$. By Corollary 2.3 and (2.3) it follows that for any $\sigma > 0$ and $j \ge 1$

$$|L^n\varphi_j(\sqrt{L})(x,y)| = |L^n\varphi(2^{-j}\sqrt{L})(x,y))| \le \frac{c_\sigma 2^{2jn}}{|B(y,2^{-j})|(1+2^j\rho(x,y))^\sigma}. \tag{5.9}$$

Exactly in the same way replacing φ with φ_0 we infer that this estimate holds for $j=0$. We choose $\sigma \ge 2n$.

Let $x \in M\setminus B_2$ and $y \in B$. Clearly, $B(z,r) \subset B(y,2r)$ and using (1.1) and $r < 2^{-j}$ we obtain

$$|B| = |B(z,r)| \le |B(y,2r)| \le c_0 2^d|B(y,r)| \le c_0 2^d |B(y,2^{-j})|.$$

This along with the obvious inequality $\rho(x,z) \le 2\rho(x,y)$ and (5.9) yield, for any $x \in M\setminus B_2$,

$$|\varphi_j(\sqrt{L})a(x)| = \Big|\int_B L^n\varphi(2^{-j}\sqrt{L})(x,y)b(y)d\mu(y)\Big|$$
$$\le \|b\|_{L^2}\|L^n\varphi(2^{-j}\sqrt{L})(x,\cdot)\|_{L^\infty(B)}|B|^{1/2} \le \frac{c|B|^{-1/p}(2^jr)^{2n}}{(1+2^j\rho(x,z))^\sigma}.$$

By Definition 5.1 we have $n > d/2p$. Choose $\varepsilon > 0$ so that $p(4n-\varepsilon)/2 > d$. Then, from above

$$\sum_{2^j<1/r} |\varphi_j(\sqrt{L})a(x)|^2 \le c|B|^{-2/p}\sum_{2^j<1/r}\frac{(2^jr)^{4n}}{(1+2^j\rho(x,z))^{4n-\varepsilon}}$$
$$\le c|B|^{-2/p}\sum_{2^j<1/r}\frac{(2^jr)^{4n}}{(2^jr)^{4n-\varepsilon}\big(1+\frac{\rho(x,z)}{r}\big)^{4n-\varepsilon}}$$
$$\le \frac{c|B|^{-2/p}}{\big(1+\frac{\rho(x,z)}{r}\big)^{4n-\varepsilon}}\sum_{2^j<1/r}(2^jr)^\varepsilon$$
$$\le \frac{c|B|^{-2/p}}{\big(1+\frac{\rho(x,z)}{r}\big)^{4n-\varepsilon}}, \quad x \in M\setminus B_2.$$

This implies

$$\Big\|\Big(\sum_{2^j<1/r}|\varphi_j(\sqrt{L})a(\cdot)|^2\Big)^{1/2}\Big\|^p_{L^p(M\setminus B_2)} \le c|B|^{-1}\int_M \frac{d\mu(x)}{\big(1+\frac{\rho(x,z)}{r}\big)^{p(4n-\varepsilon)/2}} \le c.$$

For the last inequality we used (2.4) and that $p(4n-\varepsilon)/2 > d$. Putting together the above estimates we arrive at (5.6).

Consider now the case when a is an atom of type A. Then $\operatorname{supp} a \subset B$, where $B = B(z,r)$ for some $z \in M$ and $r \ge 1$, and $\|a\|_{L^2} \le |B|^{1/2-1/p}$. In this case, we proceed exactly as above with one important distinction. As $r \ge 1$ the set of all $j \ge 0$ such that $2^j < 1/r$ is empty and, therefore, the estimate $\|a\|_{L^2} \le |B|^{1/2-1/p}$ is sufficient to obtain the same result. This completes the proof of Lemma 5.5. $\square$

Assume $f \in H^p_A$. Then there exist atoms $\{a_k\}_{k\ge1}$ (see Definition 5.1) such that $f = \sum_k \lambda_k a_k$ (with convergence in $\mathcal{D}'$) and $\sum_k |\lambda_k|^p \le 2\|f\|^p_{H^p_A}$. By the properties of φ_j it follows that $\varphi_j(\sqrt{L})f(x) = \sum_k \lambda_k\varphi_j(\sqrt{L})a_k(x)$, $x \in M$, $j \ge 0$. Therefore, with the notation (as above) $Tf := \Big(\sum_{j\ge0}|\varphi_j(\sqrt{L})f(\cdot)|^2\Big)^{1/2}$ we have for $x \in M$

$$Tf(x) = \Big\|\Big(\sum_k \lambda_k\varphi_j(\sqrt{L})a_k(x)\Big)\Big\|_{\ell^2} \le \sum_k |\lambda_k|\big\|\big(\varphi_j(\sqrt{L})a_k(x)\big)\big\|_{\ell^2} = \sum_k |\lambda_k| Ta_k(x).$$

Using the above and Lemma 5.5 we obtain

$$\|f\|^p_{F^0_{p2}} = \|Tf\|^p_p \le \sum_k |\lambda_k|^p\|Ta_k\|^p_p \le c\sum_k|\lambda_k|^p \le c\|f\|^p_{H^p_A}$$

as claimed. This completes the first part of the proof.

Assume $f \in F^0_{p2}$. We shall show that $f \in H^p_A$ and $\|f\|_{H^p_A} \le c\|f\|_{F^0_{p2}}$. To this end for the given $0 < p \le 1$ we set $s_0 = 0$, $p_0 = p$, $p_1 = 2$, and $q_0 = 2$, and impose on the parameters K, N in the construction of $\{\theta_\xi\}_{\xi\in\mathcal{X}}$ and Theorem 4.2 the additional conditions

$$K \ge 3d/2 + 1 \quad \text{and} \quad N \ge 2K + 4n + 3d + 2, \tag{5.10}$$

where $n := \lfloor d/2p\rfloor + 1$ as in Definition 5.1. Then for sufficiently small ε in the construction of $\{\theta_\xi\}$ Theorem 4.2 remains valid with B^s_{pq}, b^s_{pq} replaced by F^0_{p2}, f^0_{p2}. In particular, denoting $\mathcal{X}' := \cup_{j\ge1}\mathcal{X}_j$, we have

$$f = \sum_{\xi\in\mathcal{X}}\langle f,\tilde\theta_\xi\rangle\theta_\xi = \sum_{\xi\in\mathcal{X}_0}\langle f,\tilde\theta_\xi\rangle\theta_\xi + \sum_{\xi\in\mathcal{X}'}\langle f,\tilde\theta_\xi\rangle\theta_\xi =: f_0 + f_1, \tag{5.11}$$

where the convergence is unconditional in F^0_{p2}, and

$$\begin{aligned}\|f\|_{F^0_{p2}} &\sim \Big\|\Big(\sum_{\xi\in\mathcal{X}}\big[|\langle f,\tilde\theta_\xi\rangle|\tilde{\mathbb{1}}_{A_\xi}(\cdot)\big]^2\Big)^{1/2}\Big\|_{L^p}\\ &\sim \Big(\sum_{\xi\in\mathcal{X}_0}\big[|\langle f,\tilde\theta_\xi\rangle||A_\xi|^{1/p-1/2}\big]^p\Big)^{1/p} + \Big\|\Big(\sum_{\xi\in\mathcal{X}'}\big[|\langle f,\tilde\theta_\xi\rangle|\tilde{\mathbb{1}}_{A_\xi}(\cdot)\big]^2\Big)^{1/2}\Big\|_{L^p}.\end{aligned} \tag{5.12}$$

We split the atomic decomposition of f into two steps by decomposing first f_0 and then f_1 (see (5.11)).

From Lemma 5.3 we know that there exists a constant $c_* > 0$ such that for any $\xi \in \mathcal{X}_0$ the function $a_\xi := c_*|B(\xi,1)|^{1/2-1/p}\theta_\xi$ is an atom (of type A). On the other hand, from the definition of $\{A_\xi\}$ in §2.3 it follows that $|A_\xi| \sim |B(\xi,1)|$, $\xi \in \mathcal{X}_0$.

Setting $\lambda_\xi := c_*^{-1}\langle f, \tilde{\theta}_\xi\rangle |B(\xi,1)|^{1/p-1/2}$ we get $|\lambda_\xi| \le c|\langle f, \tilde{\theta}_\xi\rangle||A_\xi|^{1/p-1/2}$, $\xi \in \mathcal{X}_0$. From this and (5.12) we infer

$$f_0 = \sum_{\xi\in\mathcal{X}_0} \langle f, \tilde{\theta}_\xi\rangle \theta_\xi = \sum_{\xi\in\mathcal{X}_0} \lambda_\xi a_\xi \quad \text{and} \quad \sum_{\xi\in\mathcal{X}_0} |\lambda_\xi|^p \le c\|f\|^p_{F^0_{p2}}. \tag{5.13}$$

We now turn to the atomic decomposition of f_1. By (4.7) we have $\operatorname{supp}\theta_\xi \subset B_\xi$, where $B_\xi := B(\xi, \delta_j)$, $\delta_j := \tilde{c}Rb^{-j}$ for $\xi \in \mathcal{X}_j$. Denote briefly $\alpha_\xi := \langle f, \tilde{\theta}_\xi\rangle$. We may assume that $\alpha_\xi \ne 0$ for $\xi \in \mathcal{X}'$ (otherwise we remove ξ from $\mathcal{X}'$). Set

$$g(x) := \Big(\sum_{\xi\in\mathcal{X}'} |\alpha_\xi|^2 |B_\xi|^{-1} \mathbb{1}_{B_\xi}(x)\Big)^{1/2}$$

and write $\Omega_r := \{x \in M : g(x) > 2^r\}$, $r \in \mathbb{Z}$. Obviously, $\Omega_{r+1} \subset \Omega_r$ for $r \in \mathbb{Z}$ and $\cup_{r\in\mathbb{Z}}\Omega_r = \cup_{\xi\in\mathcal{X}'} B_\xi$. It is easy to see that

$$\sum_{r\in\mathbb{Z}} 2^{pr}|\Omega_r| \le c_p \int_M g(x)^p d\mu(x). \tag{5.14}$$

Indeed, we have

$$\begin{aligned}\sum_{r\in\mathbb{Z}} 2^{pr}|\Omega_r| &= \sum_{r\in\mathbb{Z}} 2^{pr} \sum_{\nu\ge r} |\Omega_\nu \setminus \Omega_{\nu+1}| = \sum_{\nu\in\mathbb{Z}} |\Omega_\nu \setminus \Omega_{\nu+1}| \sum_{r\le\nu} 2^{pr} \\ &\le c_p \sum_{\nu\in\mathbb{Z}} 2^{p\nu}|\Omega_\nu \setminus \Omega_{\nu+1}| \le c_p \sum_{\nu\in\mathbb{Z}} \int_{\Omega_\nu\setminus\Omega_{\nu+1}} g(x)^p d\mu(x) = c_p \int_M g(x)^p d\mu(x).\end{aligned}$$

Define

$$\mathcal{B}_r := \big\{B_\xi : |B_\xi \cap \Omega_r| \ge |B_\xi|/2 \;\text{ and }\; |B_\xi \cap \Omega_{r+1}| < |B_\xi|/2\big\}$$

and observe that $\mathcal{B}_r \cap \mathcal{B}_s = \emptyset$ if $r \ne s$ and $\{B_\xi\}_{\xi\in\mathcal{X}'} = \cup_{r\in\mathbb{Z}}\mathcal{B}_r$. We next introduce a partial order on the set $\{B_\xi\}$. Namely, we write $B_\eta \prec B_\xi$ if

(i) $B_\xi, B_\eta \in \mathcal{B}_r$ for some $r \in \mathbb{Z}$, and

(ii) if $\xi \in \mathcal{X}_j$, $\eta \in \mathcal{X}_k$ for some $j < k$, there exists a chain $B_{\xi_1}, \dots, B_{\xi_m} \in \mathcal{B}_r$ such that $B_{\xi_1} = B_\xi$, $B_{\xi_m} = B_\eta$, $B_{\xi_\nu} \cap B_{\xi_{\nu+1}} \ne \emptyset$ and $\operatorname{level}(\xi_\nu) < \operatorname{level}(\xi_{\nu+1})$ for $1 \le \nu \le m-1$.

Denote by $\mathcal{M}(\mathcal{B}_r)$ the set of all maximal elements $B_\xi \in \mathcal{B}_r$ with respect to $\prec$ and for each $B_\xi \in \mathcal{M}(\mathcal{B}_r)$ set $\mathcal{T}_\xi := \{B_\eta \in \mathcal{B}_r : B_\eta \prec B_\xi\}$. By assigning each ball $B_\eta \in \mathcal{B}_r$ to only one $\mathcal{T}_\xi$ we may assume that these are disjoint sets. Therefore, we have the following decomposition into disjoint "trees":

$$\{B_\eta\}_{\eta\in\mathcal{X}'} = \cup_{r\in\mathbb{Z}} \cup_{B_\xi\in\mathcal{M}(\mathcal{B}_r)} \mathcal{T}_\xi.$$

We associate with each such "tree" $\mathcal{T}_\xi$ ($\xi \in \mathcal{X}_j$) the function $f_\xi := \sum_{\eta\in\mathcal{T}_\xi} \alpha_\eta\theta_\eta$ and set

$$a_\xi := c_\star|B(\xi,3\delta_j)|^{-1/p}2^{-r}f_\xi,\; b_\xi := L^{-n}a_\xi, \text{ and } \lambda_\xi := c_\star^{-1}|B(\xi,3\delta_j)|^{1/p}2^r. \tag{5.15}$$

We next show that a_ξ is an atom if the constant $c_\star > 0$ is selected sufficiently small.

Observe first that each ball $B_\eta \in \mathcal{T}_\xi$ ($\xi \in \mathcal{X}_j$) is connected to $B_\xi := B(\xi, \delta_j)$ by a chain of balls and hence $B_\eta \subset B(\xi, \gamma)$ with $\gamma := \delta_j(1 + \sum_{\nu\ge1} 2b^{-\nu}) \le 3\delta_j$, using that $b \ge 2$. On the other hand, from the proof of Lemma 5.3 $\operatorname{supp} L^{-m}\theta_\eta \subset B_\eta$ for $0 \le m \le n$, and hence $\operatorname{supp} L^k b_\xi \subset \cup_{\eta\in\mathcal{T}_\xi} B_\eta \subset B(\xi, 3\delta_j)$, $0 \le k \le n$. Thus, to prove that a_ξ is an atom, it remains to show that if the constant $c_\star$ is sufficiently small, then

$$\|L^k b_\xi\|_{L^2} \le (3\delta_j)^{2(n-k)}|B(\xi,3\delta_j)|^{1/2-1/p}, \quad 0 \le k \le n, \tag{5.16}$$

which is equivalent to

$$\|L^{-m}a_\xi\|_{L^2} \le (3\delta_j)^{2m}|B(\xi,3\delta_j)|^{1/2-1/p}, \quad 0 \le m \le n. \tag{5.17}$$

For this we need the following Bessel type property of $\{L^{-m}\theta_\eta\}$:

Lemma 5.6. *For any sequence of numbers $\{\beta_\eta\}_{\eta\in\mathcal{X}'}$ and $0 \le m \le n$ we have*

$$\Big\|\sum_{\eta\in\mathcal{X}'}\beta_\eta L^{-m}\theta_\eta\Big\|_{L^2}^2 \le c\sum_{\eta\in\mathcal{X}'} b^{-4mj_\eta}|\beta_\eta|^2. \tag{5.18}$$

Here, j_η is the level of η, i.e. $\eta \in \mathcal{X}_{j_\eta}$.

Proof. To prove the above inequality we shall show that the elements of the Gram matrix of $\{L^{-m}\theta_\eta\}$ decay sufficiently fast away from the main diagonal, namely, if $\xi \in \mathcal{X}_j$, $\eta \in \mathcal{X}_\ell$, $\ell \ge j \ge 1$, and $0 \le m \le n$ then

$$|\langle L^{-m}\theta_\xi, L^{-m}\theta_\eta\rangle| \le cb^{-4mj}b^{-(\ell-j)N/2}\big(1+b^j\rho(\xi,\eta)\big)^{-K}. \tag{5.19}$$

To prove (5.19) we proceed similarly as in the proof of Theorem 4.5. From (4.6) we obtain

$$|\langle L^{-m}\theta_\xi, L^{-m}\theta_\eta\rangle| \le c|B(\xi,b^{-j})|^{1/2}|B(\eta,b^{-\ell})|^{1/2}|L^{-2m}\Theta(b^{-j}\sqrt{L})\Theta(b^{-\ell}\sqrt{L})(\xi,\eta)|.$$

Set $F(\lambda) := \lambda^{-4m}\Theta(\lambda)\Theta(b^{-(\ell-j)}\lambda)$. Then

$$F(b^{-j}\sqrt{L}) = b^{4mj}L^{-2m}\Theta(b^{-j}\sqrt{L})\Theta(b^{-\ell}\sqrt{L}), \tag{5.20}$$

and by Proposition 4.1 we obtain for $\nu = 0,1,\dots,K$

$$|F^{(\nu)}(\lambda)| \le \frac{cb^{-(\ell-j)N}\lambda^{2N}}{\lambda^{4m}(1+\lambda)^{2N}(1+b^{-(\ell-j)}\lambda)^{2N}} \le \frac{cb^{-(\ell-j)N/2}}{(1+\lambda)^{N/2}}, \quad \lambda \ge 1,$$

and

$$|F^{(\nu)}(\lambda)| \le \frac{cb^{-(\ell-j)N}\lambda^{2N}}{\lambda^{4m+K}(1+\lambda)^{2N}(1+b^{-(\ell-j)}\lambda)^{2N}} \le \frac{cb^{-(\ell-j)N/2}}{(1+\lambda)^{N/2}}, \quad 0 < \lambda < 1.$$

Here we used that $2N > K+4n$ and for the same reason $F^{(\nu)}(0) = 0$, $\nu = 0,\dots,K$. Now, we apply Theorem 2.4 using that $N/2 \ge K+d+1$ (see (5.10)) and obtain

$$|F(b^{-j}\sqrt{L})(x,y)| \le \frac{cb^{-(\ell-j)N/2}}{|B(x,b^{-j})|^{1/2}|B(y,b^{-j})|^{1/2}(1+b^j\rho(x,y))^K}.$$

This along with (5.20) implies (5.19).

Denote briefly $\upsilon_\xi(x) := b^{2mj}L^{-m}\theta_\xi(x)$ for $\xi \in \mathcal{X}_j$. Then, if $\xi \in \mathcal{X}_j$, $\eta \in \mathcal{X}_\ell$, $\ell \ge j \ge 1$, then

$$|\langle \upsilon_\xi, \upsilon_\eta\rangle| \le cb^{-(\ell-j)(N/2-2m)}\big(1+b^j\rho(\xi,\eta)\big)^{-K} \le cb^{-(\ell-j)(\frac{3d}{2}+1)}\big(1+b^j\rho(\xi,\eta)\big)^{-\frac{3d}{2}-1},$$

where we used that $N/2 \ge 2n+3d/2+1$ and $K \ge 3d/2+1$. This and Definition 4.3 imply that the Gram matrix $G := (\langle \upsilon_\xi, \upsilon_\eta\rangle)_{\xi,\eta\in\mathcal{X}'}$ is almost diagonal for $f_{22}^0 = \ell^2$ and by Theorem 4.4 the operator associated with this matrix is bounded on ℓ^2. Therefore, for any sequence of numbers $\{\beta_\eta\}_{\eta\in\mathcal{X}'}$ and $0 \le m \le n$,

$$\begin{aligned}\Big\|\sum_{\eta\in\mathcal{X}'}\beta_\eta L^{-m}\theta_\eta\Big\|_{L^2}^2 &= \Big\|\sum_{\eta\in\mathcal{X}'} b^{-2mj_\eta}\beta_\eta\upsilon_\eta\Big\|_{L^2}^2 \\ &\le \|G\|_{2\to 2}\sum_{\eta\in\mathcal{X}'}|b^{-2mj_\eta}\beta_\eta|^2 \le c\sum_{\eta\in\mathcal{X}'}|b^{-2mj_\eta}\beta_\eta|^2,\end{aligned}$$

which verifies (5.18). $\square$

We are now prepared to prove (5.17). Let $\xi \in \mathcal{X}_j$, $B_\xi \in \mathcal{M}(\mathcal{B}_r)$ for some $r > 0$, and $0 \le m \le n$. Then using (5.18) we get

$$\|L^{-m} f_\xi\|_{L^2}^2 = \Big\| \sum_{\eta \in \mathcal{T}_\xi} \alpha_\eta L^{-m} \theta_\eta \Big\|_{L^2}^2 \le c b^{-4mj} \sum_{\eta \in \mathcal{T}_\xi} |\alpha_\eta|^2. \tag{5.21}$$

On the other hand, for any $B_\eta \in \mathcal{T}_\xi$ we have $B_\eta \subset B(\xi, 3\delta_j)$, which gives

$$1 \le 2|B_\eta|^{-1} |B_\eta \setminus \Omega_{r+1}| = 2|B_\eta|^{-1} \int_{B(\xi,3\delta_j)\setminus\Omega_{r+1}} \mathbb{1}_{B_\eta} d\mu.$$

Thus,

$$\begin{aligned} \sum_{\eta \in \mathcal{T}_\xi} |\alpha_\eta|^2 &\le 2 \int_{B(\xi,3\delta_j)\setminus\Omega_{r+1}} \sum_{\eta \in \mathcal{T}_\xi} |\alpha_\eta|^2 |B_\eta|^{-1} \mathbb{1}_{B_\eta} d\mu \\ &\le 2 \int_{B(\xi,3\delta_j)\setminus\Omega_{r+1}} |g(x)|^2 d\mu(x) \le c|B(\xi,3\delta_j)| 2^{2r}. \end{aligned}$$

This coupled with (5.21) implies

$$\begin{aligned} \|L^{-m} a_\xi\|_{L^2} &= c_\star |B(\xi,3\delta_j)|^{-1/p} 2^{-r} \|L^{-m} f_\xi\|_{L^2} \\ &\le c c_\star b^{-2mj} |B(\xi,3\delta_j)|^{1/2-1/p} \le c c_\star (3\delta_j)^{2m} |B(\xi,3\delta_j)|^{1/2-1/p}. \end{aligned}$$

Choosing $c_\star$ so that $cc_\star = 1$ we arrive at $\|L^{-m} a_\xi\|_{L^2} \le (3\delta_j)^{2m} |B(\xi,3\delta_j)|^{1/2-1/p}$. Therefore, with this choice of $c_\star$ the function a_ξ from (5.15) is an atom.

By assumption $f \in F^0_{p2}$ and hence the representation (5.11) is valid, where the convergence is unconditional in F^0_{p2}. As F^0_{p2} is continuously embedded in $\mathcal{D}'$ [12, Prposition 7.3], the series in (5.11) converges unconditionally in $\mathcal{D}'$ as well. Thus, we can rearrange the terms in the representation of f_1 as we please, in particular,

$$f_1 = \sum_{r \in \mathbb{Z}} \sum_{B_\xi \in \mathcal{M}(\mathcal{B}_r)} \lambda_\xi a_\xi \quad \text{in } \mathcal{D}'. \tag{5.22}$$

Now, using (5.14)-(5.15) and the fact that each a_ξ, when $B_\xi \in \mathcal{M}(\mathcal{B}_r)$, is an atom we obtain

$$\begin{aligned} \sum_{r \in \mathbb{Z}} \sum_{B_\xi \in \mathcal{M}(\mathcal{B}_r)} |\lambda_\xi|^p &\le c c_\star^{-p} \sum_{r \in \mathbb{Z}} \sum_{B_\xi \in \mathcal{M}(\mathcal{B}_r)} 2^{pr} |B_\xi| \\ &\le c \sum_{r \in \mathbb{Z}} \sum_{B_\xi \in \mathcal{M}(\mathcal{B}_r)} 2^{pr} |B_\xi \cap \Omega_r| \\ &\le c \sum_{r \in \mathbb{Z}} 2^{pr} |\Omega_r| \le c\|g\|_p^p \le c\|f\|_{F^0_{p2}}^p. \end{aligned} \tag{5.23}$$

Here for the last inequality we used that $\mathbb{1}_{B_\xi}(x) \le c\mathcal{M}_1 \mathbb{1}_{A_\xi}(x)$, $\xi \in \mathcal{X}'$, and applied the maximal inequality (2.40).

From (5.13) and (5.22)-(5.23) it follows that $f \in H^p_A$ and $\|f\|_{H^p_A} \le c\|f\|_{F^0_{p2}}$. This completes the proof of Theorem 5.4. $\square$

6. Frames with small supports on the interval, ball, and simplex

The purpose of this section is to illustrate our heat kernel based method for construction of frames in three classical cases (interval, ball, and simplex), where orthogonal polynomials appear naturally as eigenfunctions; the case on the sphere is completed in [20, 16].

6.1. **Frames with small shrinking supports on $[-1,1]$ with Jacobi weights.** Consider the case when $M=[-1,1]$, $d\mu(x)=w_{\alpha,\beta}(x)dx$, where

$$w_{\alpha,\beta}(x)=w(x):=(1-x)^{\alpha}(1+x)^{\beta},\quad \alpha,\beta>-1,$$

and

$$Lf(x)=-\frac{[w(x)a(x)f'(x)]'}{w(x)},\quad a(x)=(1-x^2).$$

As is well known [24] the (normalized) Jacobi polynomials P_k, $k=0,1,\dots$, are eigenfunctions of the operator L, i.e. $LP_k=\lambda_kP_k$ with $\lambda_k=k(k+\alpha+\beta+1)$.

It is not hard to see that the operator L is essentially self-adjoint and positive. In [2] it is shown that L generates a complete strictly local Dirichlet space with an intrinsic metric on $[-1,1]$ defined by

$$\rho(x,y)=|\arccos x-\arccos y|. \tag{6.1}$$

The doubling property of the measure $d\mu$ follows readily by the following estimates on $|B(x,r)|=\mu(B(x,r))$:

$$|B(x,r)|\sim r(1-x+r^2)^{\alpha+1/2}(1+x+r^2)^{\beta+1/2}.$$

The Poincaré inequality holds true and appears in the form: For any weakly differentiable function $f:[-1,1]\to\mathbb{C}$ and an interval $I=[a,b]\subset[-1,1]$

$$\int_I|f(x)-f_I|^2w(x)dx\le c(\operatorname{diam}_\rho(I))^2\int_I|f'(x)|^2(1-x^2)w(x)dx \tag{6.2}$$

where $\operatorname{diam}_\rho(I)=\arccos a-\arccos b$, $f_I=\frac{1}{w(I)}\int_I f(x)w(x)dx$, $w(I)=\int_I w(x)dx$. We refer the reader to [2] for details and proofs.

Thus we are in a situation which fits in the general setting of complete strictly local Dirichlet spaces, where the local Poincaré inequality and doubling condition on the measure are obeyed (see [2]). The heat kernel associated with the Jacobi operator takes the form

$$p_t(x,y)=\sum_{k\ge0}e^{-\lambda_kt}P_k(x)P_k(y),\quad \lambda_k=k(k+\alpha+\beta+1), \tag{6.3}$$

and the general theory leads to Gaussian bounds on $p_t(x,y)$: For any $0<t\le1$ and $x,y\in[-1,1]$

$$\frac{c_1'\exp\{-\frac{c_1\rho^2(x,y)}{t}\}}{\big(|B(x,\sqrt t)||B(y,\sqrt t)|\big)^{1/2}}\le p_t(x,y)\le\frac{c_2'\exp\{-\frac{c_2\rho^2(x,y)}{t}\}}{\big(|B(x,\sqrt t)||B(y,\sqrt t)|\big)^{1/2}}. \tag{6.4}$$

In turn, the upper bound above implies that the finite speed propagation property holds and as a consequence we arrive at the following fundamental property of Jacobi polynomials: If f is even, $\operatorname{supp}\hat f\subset[-A,A]$ for some $A>0$, and $\hat f\in W^2_\infty$, i.e. $\|\hat f^{(2)}\|_\infty<\infty$, then for $\delta>0$ and $x,y\in[-1,1]$

$$\sum_{k\ge0}f(\delta\sqrt{\lambda_k})P_k(x)P_k(y)=0\quad\text{if}\quad\rho(x,y)>\tilde c\delta A. \tag{6.5}$$

In this case the eigenspaces have the polynomial property (the product of two polynomials of degree n is a polynomial of degree $2n$) and, therefore, the "simple" scheme from §5.3 in [2] or §4.4 in [12] produces a frame $\{\psi\}_{\xi\in\mathcal{X}}$, which can be used for decomposition of weighted Besov and Triebel-Lizorkin spaces on $[-1, 1]$ in the form

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi, \tag{6.6}$$

and the B- and F-norms of f are characterized by respective sequence norms of $\{\langle f, \psi_\xi \rangle\}$ just as in Theorems 2.10-2.11 above.

Now, the scheme from §3.3, §4 produces a pair of dual frames $\{\theta_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde{\theta}_\xi\}_{\xi\in\mathcal{X}}$ which can be used for decomposition of the B- and F-spaces with frame characterization of the norms as in Theorem 4.2. Here $\mathcal{X}$ has a multilevel structure: $\mathcal{X} = \cup_{j\ge 0}\mathcal{X}_j$ and the frame elements $\{\theta_\xi\}$ have shrinking supports, namely, $\operatorname{supp}\theta_\xi \subset B(\xi, cb^{-j})$, $\xi \in \mathcal{X}_j$, $j \ge 0$.

Remark. Here we have an example where the general method presented in this paper allows to improve on well known results and produce new results in a concrete classical setting. The Gaussian bounds (6.4) for the heat kernel (6.3) were established in [2] and also independently in [21] in the case when $\alpha, \beta \ge -1/2$. The finite speed propagation property and its important consequence (6.5) to the best of our knowledge appear explicitly first in this article and implicitly in [12]. Frames as in (6.6) and their utilization for decomposition of weighted Besov and Triebel-Lizorkin spaces on $[-1, 1]$ with weight $\omega_{\alpha,\beta}(x)$ are developed in [18] under the condition $\alpha, \beta > -1/2$, while above we assume $\alpha, \beta > -1$. Up to now frames with small shrinking supports on $[-1, 1]$ with weight $\omega_{\alpha,\beta}(x)$ were only possible from [17] in the case when $\alpha = \beta$, α is a half integer, and $\alpha \ge -1/2$, while here we operate under the assumption $\alpha, \beta > -1$. Therefore, as a whole the proposed heat kernel based development of Jacobi frames is more complete.

6.2. **Heat kernel and frames with small shrinking supports on the ball.** Let M be the unit ball $\mathbb{B}^d := \{x \in \mathbb{R}^d : |x| < 1\}$ in $\mathbb{R}^d$ and the measure be $d\nu(x) = w_\mu(x)dx$, where

$$w_\mu(x) := (1 - |x|^2)^{\mu-1/2}, \quad \mu > -1/2.$$

Here $|x|$ is the Euclidean norm of $x \in \mathbb{R}^d$. Consider the differential operator

$$L_\mu := -\Delta + \sum_{i=1}^{d}\sum_{j=1}^{d} x_i x_j \partial_i \partial_j + (2\mu + d)\sum_{j=1}^{d} x_i \partial_i, \tag{6.7}$$

which has orthogonal polynomials on $\mathbb{B}^d$ with weight w_μ as eigenfunctions. To be more specific, denote by V_n^d the space for all polynomials of degree n in d variables which are orthogonal to lower degree polynomials in $L^2(\mathbb{B}^d, w_\mu)$. These are eigenspaces of the differential operator L_μ (see e.g. [3, 5]), i.e.

$$L_\mu P = \lambda_n P \quad \text{for } P \in V_n^d \quad \text{with } \lambda_n = n(n + d + 2\mu - 1).$$

In [13] it is shown that the operator L_μ from (6.7) is essentially self-adjoint and positive, and L_μ generates a complete strictly local Dirichlet space with an intrinsic metric on $\mathbb{B}^d$ defined by

$$\rho(x, y) := \arccos\left\{\langle x, y\rangle + \sqrt{1 - |x|^2}\sqrt{1 - |y|^2}\right\}.$$

More importantly, the respective local scale-invariant Poincaré inequality holds. Furthermore, it is easy to see that $\nu(B(x, r)) \sim r^d(r^2 + 1 - |x|^2)$, which implies the validity of the doubling condition on the measure.

Therefore, the machinery of Dirichlet spaces applies and the results from [2, 12] and the current article apply in full. We next describe their main implications. If $\{P_\alpha\}_{|\alpha|=n}$ is an orthonormal basis for the space V_n^d, then the kernel of the the orthogonal projector $\mathrm{Proj}_n : L^2(\mathbb{B}^d, w_\mu) \mapsto V_n^d$ can be written in the form $P_n(w_\mu; x, y) = \sum_{|\alpha|=n} P_\alpha(x)P_\alpha(y)$; it is independent of the particular selection of the basis $\{P_\alpha\}_{|\alpha|=n}$ in V_n^d. The associated heat kernel takes the form

$$p_t(x,y) = \sum_{n\ge 0} e^{-\lambda_n t} P_n(w_\mu; x, y). \tag{6.8}$$

The Poincaré inequality and doubling property of the measure yield Gaussian bounds on the heat kernel $p_t(x,y)$, which appear just as in (6.4). In turn, the Gaussian upper bound implies the finite speed propagation property which implies the following property: If f is even, $\operatorname{supp} \hat f \subset [-A, A]$ for some $A > 0$, and $\hat f \in W_\infty^2$, then for $\delta > 0$ and $x, y \in \mathbb{B}^d$

$$\sum_{n\ge 0} f(\delta\sqrt{\lambda_n}) P_n(w_\mu; x, y) = 0 \quad \text{if} \quad \rho(x,y) > \tilde c \delta A. \tag{6.9}$$

The polynomial property of the eigenspaces V_n^d allows to apply the "simple" scheme from [2] or [12] and construct a frame $\{\psi\}_{\xi\in\mathcal{X}}$ for the weighted Besov and Triebel-Lizorkin spaces just as in Theorems 2.10-2.11 above. Furthermore, the scheme from §3.3, §4 enables us to construct a pair of dual frames $\{\theta_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde\theta_\xi\}_{\xi\in\mathcal{X}}$ for the weighted B- and F-spaces with weight $w_\mu(x)$. The supports of the frame elements $\{\theta_\xi\}_{\xi\in\mathcal{X}}$ shrink, more precisely, $\operatorname{supp} \theta_\xi \subset B(\xi, cb^{-j})$, $\xi \in \mathcal{X}_j$, $j \ge 0$.

Remark. The weighted Besov and Triebel-Lizorkin spaces on $\mathbb{B}^d$ with weight $w_\mu(x)$ and their frame decomposition have already been developed in [19] under the condition $\mu \ge 0$. Frames with small shrinking supports for the same spaces are developed in [17] under the condition that μ is a half-integer and $\mu \ge 0$. The main points in our development on the ball are that, first, with the use of the heat kernel technology from [2, 12, 13] we free the development of spaces and frames on the ball from the restriction $\mu \ge 0$, replacing it by $\mu > -1/2$, second, we develop here frames with small shrinking supports under the natural condition $\mu > -1/2$, and third, we have a characterization of atomic Hardy spaces on the ball.

6.3. **Heat kernel and frames with small supports on the simplex.** We consider now the case when M is the simplex

$$\mathbb{T}^d = \{x \in \mathbb{R}^d : x_1 \ge 0, \dots, x_d \ge 0, 1 - \|x\|_1 \ge 0\}, \quad \|x\|_1 := |x_1| + \ldots + |x_d|,$$

with measure $d\nu(x) := w_\kappa(x)dx$, where

$$w_\kappa(x) = x_1^{\kappa_1 - \frac12} \cdots x_d^{\kappa_d - \frac12} (1 - \|x\|_1)^{\kappa_{d+1} - \frac12}, \quad \kappa_i > -1/2.$$

Consider the differential operator (with $\partial_i := \partial/\partial x_i$ and $|\kappa| := \kappa_1 + \cdots + \kappa_d$)

$$L_\kappa := -\sum_{i=1}^d x_i \partial_i^2 + \sum_{i=1}^d \sum_{j=1}^d x_i x_j \partial_i \partial_j - \sum_{i=1}^d \big(\kappa_i + \tfrac12 - (|\kappa| + \tfrac{d+1}{2}) x_i\big)\partial_i.$$

As is well known (see [3]) the orthogonal polynomials on $\mathbb{T}^d$ with respect to w_κ are eigenfunctions of L_κ. More explicitly, if V_n^d denotes the space of all polynomials of degree n which are orthogonal to lower degree polynomials in $L^2(\mathbb{T}^d, w_\kappa)$, then

$$L_\kappa P = \lambda_n P \quad \text{for} \quad P \in V_n^d \quad \text{with} \quad \lambda_n := n(n+2l_\kappa), \;\; l_\kappa := |\kappa| + \frac{d-1}{2}.$$

In [13] it is shown that L_κ is an essentially self-adjoint positive operator which generates a complete strictly local Dirichlet space with an intrinsic metric on $\mathbb{T}^d$ defined by

$$\rho(x,y) := \arccos\Big\{\sqrt{x_1 y_1} + \cdots + \sqrt{x_d y_d} + \sqrt{(1-\|x\|_1)(1-\|y\|_1)}\Big\}.$$

Moreover, the respective local scale-invariant Poincaré inequality is valid [13]. Also, it is easy to see that

$$|B(x,r)| = \nu(B(x,r)) \sim r^d \prod_{i=1}^{d+1} (r^2 + x_i)^{\kappa_i},$$

which implies the doubling property of the measure. Thus, we are again in a position to run the machinery of Dirichlet spaces and the results from [2, 12] and the previous sections apply in full.

Assuming that $\{P_\alpha\}_{|\alpha|=n}$ is an orthonormal basis for the space V_n^d, the kernel of the orthogonal projector $\operatorname{Proj}_n : L^2(\mathbb{T}^d, w_\kappa) \mapsto V_n^d$ can be written in the form $P_n(w_\kappa; x, y) = \sum_{|\alpha|=n} P_\alpha(x) P_\alpha(y)$. Therefore, the associated heat kernel can be written as

$$p_t(x,y) = \sum_{n\ge 0} e^{-\lambda_n t} P_n(w_\kappa; x, y). \tag{6.10}$$

Gaussian bounds on $p_t(x,y)$ follow by the Poincaré inequality and the doubling property of the measure, namely, for $0 < t \le 1$ and $x, y \in \mathbb{T}^d$

$$\frac{c_1' \exp\{-\frac{c_1 \rho^2(x,y)}{t}\}}{\big(|B(x,\sqrt{t})||B(y,\sqrt{t})|\big)^{1/2}} \le p_t(x,y) \le \frac{c_2' \exp\{-\frac{c_2 \rho^2(x,y)}{t}\}}{\big(|B(x,\sqrt{t})||B(y,\sqrt{t})|\big)^{1/2}}. \tag{6.11}$$

As a consequence the finite speed propagation property is valid, and therefore, the following property holds: If f is even, $\operatorname{supp} \hat{f} \subset [-A, A]$ for some $A > 0$, and $\hat{f} \in W_\infty^2$, then for $\delta > 0$ and $x, y \in \mathbb{T}^d$

$$\sum_{n\ge 0} f(\delta\sqrt{\lambda_n}) P_n(w_\kappa; x, y) = 0 \quad \text{if} \quad \rho(x,y) > \tilde{c}\delta A. \tag{6.12}$$

For more details and proofs, see [13].

As on the interval and ball, the "simple" scheme from [2] or [12] produces a frame $\{\psi\}_{\xi\in\mathcal{X}}$ for the weighted Besov and Triebel-Lizorkin spaces on $\mathbb{T}^d$ with weight $w_\kappa(x)$ just as in Theorems 2.10-2.11 above. In turn, the scheme from §3.3 and §4 produces a pair of dual frames $\{\theta_\xi\}_{\xi\in\mathcal{X}}$, $\{\tilde{\theta}_\xi\}_{\xi\in\mathcal{X}}$ for the weighted B- and F-spaces, where $\operatorname{supp} \theta_\xi \subset B(\xi, cb^{-j})$, $\xi \in \mathcal{X}_j$, $j \ge 0$.

Remark. Note that the theory of weighted Besov and Triebel-Lizorkin spaces with full indices on $\mathbb{T}^d$ with weight $w_\kappa(x)$, $\kappa_i > -1/2$, and their frame decomposition follows by the general theory from [2, 12] and the development of the heat kernel on $\mathbb{T}^d$ in [13]. However, most of the components of this theory have already been developed in [11] in the case when $\kappa_i \ge 0$. Now, the advances in [2, 12, 13] allow to handle the general case when $\kappa_i > -1/2$. Another challenging problem that was solved using the heat kernel technology is the establishment of sharp lower bound estimates on L^p norms of frame elements and operator kernels. The development of frames on $\mathbb{T}^d$ with small shrinking supports and atomic Hardy spaces on $\mathbb{T}^d$ is entirely new.

7. Appendix

Here we carry out the proof of Theorem 4.4. For this we need two lemmas.

Lemma 7.1. *Let $0 < t \le 1$ and $M > d/t$. Then for any sequence of complex numbers $\{h_\eta\}_{\eta\in\mathcal{X}_m}$, $m \ge 0$, we have for $x \in A_\xi$, $\xi \in \mathcal{X}$,*

$$\sum_{\eta\in\mathcal{X}_m} |h_\eta|\Big(1+\frac{\rho(\xi,\eta)}{\max\{\ell(\xi),\ell(\eta)\}}\Big)^{-M} \le c_* \max\Big\{b^{(m-j)d/t},1\Big\}\, M_t\Big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\mathbb{1}_{A_\eta}\Big)(x).$$

Here the constant c_* takes the form $c_* = c_1c_2^{1/t}\delta^{-1}$ with $c_1, c_2 > 1$ constants independent of t,δ if $M \ge d/t+\delta$, $0 < \delta \le 1$.

Proof. Consider the case $\ell(\xi) \ge \ell(\eta)$. The proof in the case $\ell(\xi) < \ell(\eta)$ is similar and will be omitted. Let $\xi \in \mathcal{X}_j$ $(j \le m)$ and set $\Omega_0 := \{\eta \in \mathcal{X}_m : \rho(\eta,\xi) \le c^\diamond b^{-j}\}$ and

$$\Omega_\nu := \{\eta \in \mathcal{X}_m : c^\diamond b^{\nu-1} < b^j\rho(\eta,\xi) \le c^\diamond b^\nu\},\quad \nu \ge 1,$$

where $c^\diamond = \gamma/4$ with γ the constant from the construction of Frame # 1 in §2.3. Set

$$B_\nu := B(\xi, c^\diamond b^{-m}(1+b^{\nu-j+m})),\quad \nu\ge 0.$$

Note that $A_\eta \subset B_\nu$ if $\eta \in \Omega_\nu$ and hence $B_\nu \subset B(\eta, 2c^\diamond b^{-m}(1+b^{\nu-j+m}))$ implying

$$\begin{aligned}|B_\nu| &\le |B(\eta,2c^\diamond b^{-m}(1+b^{\nu-j+m}))|\\ &\le c(1+b^{\nu-j+m})^d|B(\eta,2^{-1}c^\diamond b^{-m})| \le cb^{(\nu-j+m)d}|A_\eta|,\end{aligned}$$

where we used (1.2) and the fact that $B(\eta,2^{-1}c^\diamond b^{-m}) \subset A_\eta \subset B(\eta,c^\diamond b^{-m})$ for $\eta \in \mathcal{X}_m$, see §2.3. Thus

$$|B_\nu|/|A_\eta| \le cb^{(\nu-j+m)d},\quad \eta\in\Omega_\nu. \tag{7.1}$$

Since $0 < t \le 1$ we have

$$\begin{aligned}\sum_{\eta\in\mathcal{X}_m}|h_\eta|\big(1+b^j\rho(\xi,\eta)\big)^{-M} &\le (2/c^\diamond)^M\sum_{\nu\ge0}b^{-\nu M}\sum_{\eta\in\Omega_\nu}|h_\eta|\\ &\le (2/c^\diamond)^M\sum_{\nu\ge0}b^{-\nu M}\Big(\sum_{\eta\in\Omega_\nu}|h_\eta|^t\Big)^{1/t}.\end{aligned}$$

From this and (7.1) we obtain for $x \in A_\xi$

$$\begin{aligned}\sum_{\eta\in\Omega_\nu}|h_\eta|^t &= \int_M\Big(\sum_{\eta\in\Omega_\nu}|h_\eta||A_\eta|^{-1/t}\mathbb{1}_{A_\eta}(y)\Big)^t d\mu(y)\\ &= \frac{1}{|B_\nu|}\int_M\Big(\sum_{\eta\in\Omega_\nu}|h_\eta|\Big(\frac{|B_\nu|}{|A_\eta|}\Big)^{1/t}\mathbb{1}_{A_\eta}(y)\Big)^t d\mu(y)\\ &\le cb^{(\nu-j+m)d}\frac{1}{|B_\nu|}\int_{B_\nu}\Big(\sum_{\eta\in\Omega_\nu}|h_\eta|\mathbb{1}_{A_\eta}(y)\Big)^t d\mu(y)\\ &\le cb^{(\nu-j+m)d}\Big[\mathcal{M}_t\Big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\mathbb{1}_{A_\eta}\Big)(x)\Big]^t.\end{aligned}$$

Therefore, since $M > d/t$ we get for $x \in A_\xi$

$$\sum_{\eta\in\mathcal{X}_m} |h_\eta|\big(1+b^j d(\xi,\eta)\big)^{-M} \le c\sum_{\nu\ge 0} b^{-\nu M} b^{(\nu-j+m)d/t}\mathcal{M}_t\Big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\mathbb{1}_{A_\eta}\Big)(x)$$
$$\le c_* b^{(m-j)d/t}\mathcal{M}_t\Big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\mathbb{1}_{A_\eta}\Big)(x),$$

where the constant c_* is of the form $c_* = c_1 c_2^M c_3^{1/t}\delta^{-1}$ if $M \ge d/t+\delta$.

If $M \ge d/t+\delta$, $0<\delta\le 1$, then everywhere above M can be replaced by $d/t+\delta$, which will result in a constant c_* of the form $c_* = c_1 c_2^{1/t}\delta^{-1}$ as claimed. $\square$

In the next lemma we specify the constants in certain discrete Hardy inequalities that will be needed.

Lemma 7.2. *Let $\gamma > 0$, $0 < q < \infty$, $b > 1$, and $a_m \ge 0$ for $m \ge 0$. Then*

$$\Big(\sum_{j\ge 0}\Big(\sum_{m\ge j} b^{-(m-j)\gamma}a_m\Big)^q\Big)^{1/q} \le c_\natural\Big(\sum_{m\ge 0} a_m^q\Big)^{1/q} \tag{7.2}$$

and

$$\Big(\sum_{j\ge 0}\Big(\sum_{m=0}^{j} b^{-(j-m)\gamma}a_m\Big)^q\Big)^{1/q} \le c_\natural\Big(\sum_{m\ge 0} a_m^q\Big)^{1/q}. \tag{7.3}$$

Above the constant $c_\natural$ is of the form $c_\natural = c_1(c_2/q)^{1/q}$, where $c_1, c_2 > 0$ are constants depending only on γ and b.

The proof of this lemma is standard and simple; we omit it.

Proof of Theorem 4.4. We shall only establish the result for the spaces $\tilde f^s_{pq}$, that is,

$$\|Ah\|_{\tilde f^s_{pq}} \le c\|A\|_\delta \|h\|_{\tilde f^s_{pq}}. \tag{7.4}$$

The proof in the other cases is similar and will be omitted.

Let A be an almost diagonal operator on $\tilde f^s_{pq}$ in the sense of Definition 4.3 with associated matrix $(a_{\xi\eta})_{\xi,\eta\in\mathcal{X}}$ and let $h \in \tilde f^s_{pq}$. As compactly supported sequences are dense in $\tilde f^s_{pq}$ $(p,q<\infty)$ we may assume that the sequence h is compactly supported. Then we have $(Ah)_\xi = \sum_{\eta\in\mathcal{X}} a_{\xi\eta}h_\eta$. By the definition of $\tilde f^s_{pq}$, we have

$$\|Ah\|_{\tilde f^s_{pq}} := \Big\|\Big(\sum_{\xi\in\mathcal{X}}\big[|A_\xi|^{-s/d}|(Ah)_\xi|\tilde{\mathbb{1}}_{A_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p}$$
$$= \Big\|\Big(\sum_{\xi\in\mathcal{X}}\big[|A_\xi|^{-s/d}\sum_{\eta\in\mathcal{X}}|a_{\xi\eta}||h_\eta|\tilde{\mathbb{1}}_{A_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p} \le c(\mathcal{Q}_1+\mathcal{Q}_2),$$

where $c = 2^{1/p+1/q}$,

$$\mathcal{Q}_1 := \Big\|\Big(\sum_{\xi\in\mathcal{X}}\big[|A_\xi|^{-s/d}\sum_{\ell(\eta)\le\ell(\xi)}|a_{\xi\eta}||h_\eta|\tilde{\mathbb{1}}_{A_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p}, \quad \text{and}$$

$$\mathcal{Q}_2 := \Big\|\Big(\sum_{\xi\in\mathcal{X}}\big[|A_\xi|^{-s/d}\sum_{\ell(\eta)>\ell(\xi)}|a_{\xi\eta}||h_\eta|\tilde{\mathbb{1}}_{A_\xi}(\cdot)\big]^q\Big)^{1/q}\Big\|_{L^p}.$$

We next estimate $\mathcal{Q}_1$. Suppose $\xi \in \mathcal{X}_j$, $\eta \in \mathcal{X}_m$, and $m \ge j$; hence $\ell(\eta) \le \ell(\xi)$. We know that $B(\xi, 2^{-1}c^\diamond b^{-j}) \subset A_\xi \subset B(\xi, c^\diamond b^{-j})$ with $c^\diamond = \gamma b^{-2}$, $0 < \gamma < 1$, and similarly for A_η. We use the above, (2.2), and (1.2) to obtain

$$\begin{aligned}|A_\xi| &\le |B(\xi, c^\diamond b^{-j})| \le c_0\Big(1 + \frac{\rho(\xi,\eta)}{c^\diamond b^{-j}}\Big)^d |B(\eta, c^\diamond b^{-j})| \\ &\le c_0^2(2/c^\diamond)^d b^{(m-j)d}\Big(1 + b^j\rho(\xi,\eta)\Big)^d |B(\eta, 2^{-1}c^\diamond b^{-m})| \\ &\le c_0^2(2/c^\diamond)^d b^{(m-j)d}\Big(1 + b^j\rho(\xi,\eta)\Big)^d |A_\eta|.\end{aligned}$$

Therefore,

$$|A_\xi| \le c_\dagger\Big(\frac{\ell(\xi)}{\ell(\eta)}\Big)^d\Big(1 + \frac{\rho(\xi,\eta)}{\ell(\xi)}\Big)^d|A_\eta|, \quad c_\dagger := c_0^2(2/c^\diamond)^d. \tag{7.5}$$

Using this and $\|A\|_\delta < \infty$ (see Definition 4.3) it readily follows that whenever $\ell(\eta) \le \ell(\xi)$

$$|a_{\xi\eta}| \le c_\flat\|A\|_\delta\Big(\frac{\ell(\eta)}{\ell(\xi)}\Big)^{\mathcal{J}+\delta}\Big(\frac{|A_\xi|}{|A_\eta|}\Big)^{s/d+1/2}\Big(1 + \frac{\rho(\xi,\eta)}{\ell(\xi)}\Big)^{-\mathcal{J}-\delta}, \quad c_\flat := c_\dagger^{|s|+d/2}.$$

Denote briefly $\lambda_\xi := |A_\xi|^{-s/d-1/2}\mathbb{1}_{A_\xi}(\cdot)$ and choose t so that $d/t = \mathcal{J} + \delta/2$. Then $0 < t < \min\{1, p, q\}$ and $\mathcal{J} + \delta - d/t > 0$. We have

$$\begin{aligned}\frac{\mathcal{Q}_1}{\|A\|_\delta} &\le c_\flat\Big\|\Big(\sum_{\xi\in\mathcal{X}}\Big[\sum_{\ell(\eta)\le\ell(\xi)}\Big(\frac{\ell(\eta)}{\ell(\xi)}\Big)^{\mathcal{J}+\delta}\Big(\frac{|A_\xi|}{|A_\eta|}\Big)^{s/d+1/2} \\ &\qquad\qquad \times\Big(1 + \frac{\rho(\xi,\eta)}{\ell(\xi)}\Big)^{-\mathcal{J}-\delta}|h_\eta|\lambda_\xi(\cdot)\Big]^q\Big)^{\frac{1}{q}}\Big\|_{L^p} \\ &= c_\flat\Big\|\Big(\sum_{j\ge0}\sum_{\xi\in\mathcal{X}_j}\Big[\sum_{m\ge j} b^{(j-m)(\mathcal{J}+\delta)}\sum_{\eta\in\mathcal{X}_m}\Big(\frac{|A_\xi|}{|A_\eta|}\Big)^{s/d+1/2} \\ &\qquad\qquad \times|h_\eta|\big(1 + b^j\rho(\xi,\eta)\big)^{-\mathcal{J}-\delta}\lambda_\xi(\cdot)\Big]^q\Big)^{\frac{1}{q}}\Big\|_{L^p}.\end{aligned}$$

We now apply Lemma 7.1, the Hardy inequality (7.2), and the maximal inequality (2.40) to obtain

$$\begin{aligned}\frac{\mathcal{Q}_1}{\|A\|_\delta} &\le c_*c_\flat\Big\|\Big(\sum_{j\ge0}\sum_{\xi\in\mathcal{X}_j}\Big[\sum_{m\ge j} b^{(j-m)(\mathcal{J}+\delta-d/t)} \\ &\qquad \times M_t\Big(\sum_{\eta\in\mathcal{X}_m}\Big(\frac{|A_\xi|}{|A_\eta|}\Big)^{s/d+1/2}|h_\eta|\mathbb{1}_{A_\eta}\Big)(\cdot)\lambda_\xi(\cdot)\Big]^q\Big)^{\frac{1}{q}}\Big\|_{L^p} \\ &\le c_*c_\flat\Big\|\Big(\sum_{j\ge0}\Big[\sum_{m\ge j} b^{(j-m)\delta/2}M_t\Big(\sum_{\eta\in\mathcal{X}_m}|h_\eta|\lambda_\eta\Big)\Big]^q\Big)^{\frac{1}{q}}\Big\|_{L^p} \\ &\le c_*c_\natural c_\flat\Big\|\Big(\sum_{j\ge0}M_t\Big(\sum_{\xi\in\mathcal{X}_j}|h_\xi|\lambda_\xi\Big)^q\Big)^{\frac{1}{q}}\Big\|_{L^p} \le c_*c_\natural c_\sharp c_\flat\|h\|_{\tilde{f}^s_{pq}}.\end{aligned}$$

Here $c_* = c_1c_2^{1/t}\delta^{-1}$ is from Lemma 7.1, $c_\natural = c_3(c_4/q)^{1/q}$ is from Lemma 7.2, $c_\flat = c_5^{|s|}$ and

$$c_\sharp = c_6\max\big\{p, (p/t-1)^{-1}\big\}\max\big\{1, (q/t-1)^{-1}\big\},$$

is from (2.40). It is readily seen that the constant $c = c_* c_\natural c_\sharp c_\flat$ is of the claimed form.

To estimate $\mathcal{Q}_2$ we again use that $\|A\|_\delta < \infty$ and (7.5) with the roles of ξ and η interchanged to obtain whenever $\ell(\eta) > \ell(\xi)$

$$|a_{\xi\eta}| \le c_\flat \|A\|_\delta \Big(\frac{\ell(\xi)}{\ell(\eta)}\Big)^\delta \Big(\frac{|A_\xi|}{|A_\eta|}\Big)^{s/d+1/2} \Big(1 + \frac{\rho(\xi,\eta)}{\ell(\eta)}\Big)^{-\mathcal{J}-\delta}.$$

Setting again $\lambda_\xi := |A_\xi|^{-s/d-1/2} \mathbb{1}_{A_\xi}(\cdot)$ we get

$$\begin{aligned}\frac{\mathcal{Q}_2}{\|A\|_\delta} &\le c_\flat \Big\| \Big(\sum_{\xi\in\mathcal{X}} \Big[\sum_{\ell(\eta)>\ell(\xi)} \Big(\frac{\ell(\xi)}{\ell(\eta)}\Big)^\delta \Big(\frac{|A_\xi|}{|A_\eta|}\Big)^{s/d+1/2} \\ &\qquad\qquad \times \Big(1 + \frac{\rho(\xi,\eta)}{\ell(\eta)}\Big)^{-\mathcal{J}-\delta} |h_\eta| \lambda_\xi(\cdot)\Big]^q\Big)^{\frac{1}{q}} \Big\|_{L^p} \\ &= c_\flat \Big\| \Big(\sum_{j\ge 0} \sum_{\xi\in\mathcal{X}_j} \Big[\sum_{m<j} b^{(m-j)\delta} \sum_{\eta\in\mathcal{X}_m} \Big(\frac{|A_\xi|}{|A_\eta|}\Big)^{s/d+1/2} \\ &\qquad\qquad \times |h_\eta| \big(1 + b^m \rho(\xi,\eta)\big)^{-\mathcal{J}-\delta} \lambda_\xi(\cdot)\Big]^q\Big)^{\frac{1}{q}} \Big\|_{L^p}.\end{aligned}$$

We use again Lemma 7.1, the Hardy inequality (7.3), and the maximal inequality (2.40) to obtain

$$\begin{aligned}\frac{\mathcal{Q}_2}{\|A\|_\delta} &\le c_* c_\flat \Big\| \Big(\sum_{j\ge 0} \sum_{\xi\in\mathcal{X}_j} \Big[\sum_{m<j} b^{(m-j)\delta} \\ &\qquad \times M_t \Big(\sum_{\eta\in\mathcal{X}_m} \Big(\frac{|A_\xi|}{|A_\eta|}\Big)^{s/d+1/2} |h_\eta| \mathbb{1}_{A_\eta}\Big) \lambda_\xi(\cdot)\Big]^q\Big)^{\frac{1}{q}} \Big\|_{L^p} \\ &\le c_* c_\flat \Big\| \Big(\sum_{j\ge 0} \Big[\sum_{m<j} 2^{(m-j)\delta} M_t \Big(\sum_{\eta\in\mathcal{X}_m} |h_\eta| \lambda_\eta\Big)\Big]^q\Big)^{\frac{1}{q}} \Big\|_{L^p} \\ &\le c_* c_\natural c_\flat \Big\| \Big(\sum_{j\ge 0} \Big[M_t \big(\sum_{\xi\in\mathcal{X}_j} |h_\xi| \lambda_\xi\big)\Big]^q\Big)^{\frac{1}{q}} \Big\|_{L^p} \le c_* c_\natural c_\sharp c_\flat \|h\|_{\tilde{f}^s_{pq}},\end{aligned}$$

where the constants $c_*, c_\natural, c_\sharp, c_\flat$ are as above. The above estimates for $\mathcal{Q}_1$ and $\mathcal{Q}_2$ yield (7.4). $\square$

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