2000:14

Inequalities of Duffin-Schaeffer type

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Abstract

We prove here that if an algebraic polynomial $f$ of degree at most $n$ has smaller absolute values than $T_n$ (the $n$-th Chebyshev polynomial of the first kind) at arbitrary $n + 1$ points in $[-1, 1]$, which interlace with the zeros of $T_n$, then the uniform norm of $f'$ is smaller than $n^2$. This is an extension of a classical result obtained by Duffin and Schaeffer.

1 Introduction and statement of the result

Denote by $\pi_n$ the class of algebraic polynomials of degree at most $n$, and by $\| \cdot \|$ the supremum norm in $[-1, 1]$. The classical inequality of brothers Markov [5], [6] asserts that among all $f \in \pi_n$ satisfying

$$\|f\| \leq 1$$

the Chebyshev polynomial of the first kind $T_n(x) = \cos n \arccos x$ has the greatest norm of its $k$th derivative ($k = 1, \ldots, n$). A remarkable extension of this result was found by Duffin and Schaeffer [3], who showed that this extremal property of $T_n$ persists under a weaker assumption than (1). Namely, they showed that $T_n$ still has the largest uniform norm of its $k$-th derivative in the wider class of polynomials from $\pi_n$, satisfying

$$|f(\cos (\nu \pi / n))| \leq 1, \ \nu = 0, \ldots, n$$

(2)

(actually, Duffin and Schaeffer proved a more general result, including an inequality over a strip in the complex plane, but this does not fall in the frame of the present paper). The points

$$\eta_\nu := \cos (\nu \pi / n), \ \nu = 0, \ldots, n$$

are the local extremum points for $T_n$ in $[-1, 1]$, and $|T_n(\eta_\nu)| = 1$. Thus, the result of Duffin and Schaeffer may be viewed as a comparison type theorem: the inequality $|f| \leq |T_n|$ at the points of local extrema for $T_n$ induces the inequalities $\|f^{(k)}\| \leq \|T_n^{(k)}\|$ for $k = 1, \ldots, n$. This suggests the following

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*This research was supported by ONR/ARO Research Contract DEPSCoR – DAAG55-99-1-0002
**Definition.** A polynomial $Q \in \pi_n$ and a mesh $\Delta = \{t_{\nu}\}_{\nu=0}^n$ ($1 \geq t_0 > t_1 > \ldots > t_n \geq -1$) are said to admit Duffin and Schaeffer type inequality (DS-inequality), if for every $f \in \pi_n$ the assumption $|f(t_{\nu})| \leq |Q(t_{\nu})|$ for $\nu = 0, \ldots, n$ implies $\|f'\| \leq \|Q'\|$, or, more generally, $\|f^{(k)}\| \leq \|Q^{(k)}\|$ for $k = 1, \ldots, n$.

Note that in our definition the comparison points $\{t_{\nu}\}_{\nu=0}^n$ are not necessarily assumed to be extremum points for $Q$.

In 1992 A. Shadrin [13] proposed a simple proof of Markov inequality under the assumptions (2). Based on a theorem of Shadrin, Bojanov and Nikolov [2] proved a DS-inequality for $Q = P^{(n)}_n$ the ultraspherical polynomials, when the mesh $\Delta$ consists of the local extremum points of $P^{(n)}_n$.

**Theorem A.** Let $Q := P^{(n)}_n (\lambda > -1/2)$ and $\{t_{\nu}\}_{\nu=0}^n$ be the zeros of $(1 - x^2)Q'(x)$. If $f \in \pi_n$ satisfies

$$|f(t_{\nu})| \leq |Q(t_{\nu})| \text{ for } \nu = 0, \ldots, n,$$

then

$$\|f^{(k)}\| \leq \|Q^{(k)}\|$$

for all $k \in \{1, \ldots, n\}$, if $\lambda \geq 0$, and for $k \geq 2$, if $\lambda \in (-1/2, 0)$. Equality is possible if and only if $f = cQ$ with $|c| = 1$.

The special case $\lambda = 0$ comes down to the classical inequality of Duffin and Schaeffer.

For some other DS-inequalities, we refer the reader to [4], [7], [8], [9], [10], [11]. In particular, the following result has been proved in [9]:

**Theorem B.** If $f \in \pi_n$ satisfies $|f(\pm 1)| \leq 1$ and

$$|f(x)| \leq \sqrt{1-x^2} \text{ at the zeros of } T_{n-1},$$

then

$$\|f^{(k)}\| \leq \|T^{(k)}_{n-1}\| \text{ for } k = 1, \ldots, n.$$ 

Moreover, equality is possible if and only if $f = cT_n$ with $|c| = 1$.

Theorems A and B show that for $Q = T_n$ DS-inequality holds at least for two choices of “check points”, namely, for those formed by the zeros of $(1-x^2)T'_n(x)$ and by the zeros of $(1-x^2)T_{n-1}(x)$. We naturally come to the question: What are the meshes $\Delta$ admitting DS-inequality with $Q = T_n$? The aim of this paper is to show that for $k = 1$ each mesh $\Delta = \{t_{\nu}\}_{\nu=0}^n$ whose points interlace with the zeros of $T_n$ is admissible.

**Theorem 1.** Let $\{t_{\nu}\}_{\nu=0}^n$ satisfy $1 \geq t_0 > \xi_1 > t_1 > \ldots > \xi_n > t_n \geq -1$, where $\{\xi_\nu\}_{\nu=1}^n$ are the zeros of $T_n$, i.e., $\xi_\nu = \cos ((2\nu+1)\pi/(2n))$. If $f \in \pi_n$ and

$$|f(t_{\nu})| \leq |T_{n}(t_{\nu})| \text{ for } \nu = 0, \ldots, n,$$

then

$$\|f'\| \leq n^2. \quad (3)$$

Moreover, equality in (3) is possible if and only if $f = cT_n$ with $|c| = 1$. 

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Note that the set of all admissible meshes $\Delta$ (i.e., such that DS-inequality holds with $Q = T_n$) is not substantially larger than the one described in Theorem 1. In fact, the points of any admissible mesh must separate the zeros of $T_n$ (see Section 4).

The proof of Theorem 1 relies on a pointwise inequality given by the next theorem, which was suggested to the author by A. Shadrin [15].

**Theorem 2.** Let $Q \in \pi_n$ have $n$ distinct zeros $\{x_\nu\}_{\nu=1}^n$, all located in $(-1, 1)$. Let $\{t_j\}_{j=0}^n$ satisfy $1 \geq t_0 > x_1 > t_1 > \ldots > x_n > t_n \geq -1$. If $f \in \pi_n$ and

$$|f(t_j)| \leq |Q(t_j)| \quad \text{for} \ j = 0, \ldots, n,$$

then for each $k \in \{1, \ldots, n\}$ and for every $x \in [-1, 1]$ there holds

$$|f^{(k)}(x)| \leq \max\{|Q^{(k)}(x)|, |Q^{(k)}_\nu(x)|, \nu = 1, \ldots, n\},$$

where

$$Q_\nu(x) = Q(x) \frac{1 - x_\nu x}{x - x_\nu}.$$

The paper is organized as follows. In Section 2, we summarize some results from V. Markov’s paper and prove Theorem 2. The proof of Theorem 1 is given in Section 3. Section 4 contains some concluding remarks and points out to a possible application of Theorem 1 to the estimation of the round-off error in the Lagrange differentiation formula.

## 2 Proof of Theorem 2

We start with an observation from the original work of V. Markov [6], concerning polynomial interpolation and pointwise estimates for polynomial derivatives. We formulate it in two lemmas.

**Definition.** Let $p \in \pi_n$ or $p \in \pi_{n+1}$, $q \in \pi_n$, and $p$, $q$ have only real and simple zeros, say $\{t_j\}_{j=1}^{n+1}$ and $\{t_j\}_{j=1}^n$. The zeros of $p$ and $q$ are said to interlace, if

$$t_1 \leq t_2 \leq \ldots \leq t_{n-1} \leq t_n (\leq t_{n+1}).$$

If only strict inequalities appear above, then the zeros of $p$ and $q$ are said to interlace strictly.

The first Markov’s lemma reveals a simple (and, as a matter of fact, very useful) property of the zeros of algebraic polynomials.

**Lemma 1.** Let $p$ and $q$ be algebraic polynomials ($p \neq q$), which have only real and simple zeros. If the zeros of $p$ and $q$ interlace, then the zeros of $p'$ and $q'$ interlace strictly.
A proof of Lemma 1 can be found in [12, Lemma 2.7.1], or in [13]. Note that for polynomials of the same degree the claim of Lemma 1 can be viewed as a monotone dependence of the zeros of the derivative with respect to the zeros of the polynomial ([1, p. 39]).

Given a mesh $\Delta = \{t_j\}_{j=0}^n (1 \geq t_0 > t_1 > \ldots > t_n \geq -1)$, and $\epsilon := \{\epsilon_j\}_{j=0}^n$ ($\epsilon_j > 0, j = 0, \ldots, n$), we define the set of polynomials

$$\Omega_n(\Delta, \epsilon) := \{f \in \pi_n : |f(t_j)| \leq \epsilon_j, j = 0, \ldots, n\}.$$ Clearly, $\Omega_n(\Delta, \epsilon)$ is a compact set.

Define real valued polynomials $\{P_\nu\}_{\nu=0}^n = \{P_\nu(\Delta, \epsilon; \cdot)\}_{\nu=0}^n \in \Omega_n(\Delta, \epsilon)$ by

$$|P_\nu(t_j)| = \epsilon_j \quad \text{for } j, \nu = 0, \ldots, n,$$

$$P_0(t_{j-1})P_0(t_j) < 0 \quad \text{for } j = 1, \ldots, n,$$

and, for each $\nu = 1, \ldots, n$,

$$P_\nu(t_{\nu-1})P_\nu(t_{\nu}) > 0, P_\nu(t_{\nu-1})P_\nu(t_j) < 0 \quad \text{for } j \neq \nu.$$ Evidently, the above conditions determine $\{P_\nu\}_{\nu=0}^n$ uniquely up to a multiplier $-1$. Theorem 2 follows easily from the next lemma.

**Lemma 2.** For each $x \in [-1, 1]$ and for every $k \in \{1, \ldots, n\}$,

$$\sup\{|f^{(k)}(x)| : f \in \Omega_n(\Delta, \epsilon)\} = \max\{|P_\nu^{(k)}(x)|, \nu = 0, \ldots, n\}.$$ 

**Proof.** Note first that the sup is attainable since $\Omega_n(\Delta, \epsilon)$ is a compact. Set

$$\omega(t) := (t-t_0)\ldots(t-t_n), \quad \omega_\nu(t) := \omega(t)/(t-t_\nu) \quad (\nu = 0, \ldots, n),$$

then for $f \in \Omega_n(\Delta, \epsilon)$ and a fixed $x \in [-1, 1]$ the Lagrange interpolation formula yields

$$|f^{(k)}(x)| = \left| \sum_{j=0}^n \frac{\omega_j^{(k)}(x)}{\omega_j(t_j)} f(t_j) \right| \leq \sum_{j=0}^n \left| \frac{\omega_j^{(k)}(x)}{\omega_j(t_j)} \right| \epsilon_j.$$ (4)

The upper bound is attained if $|f(t_j)| = \epsilon_j$ for $j = 0, \ldots, n$ and $f$ has a suitable sign pattern at the points $\{t_j\}$. Next, we show that the polynomials $\{P_\nu\}_{\nu=0}^n$ provide a complete set of appropriate sign patterns. For any pair of indices $i, j \in \{0, \ldots, n\}$, $i < j$ the zeros of $\omega_i$ and $\omega_j$ interlace (though not strictly), therefore, in view of Lemma 1, the zeros $\{\gamma_{i, \mu}\}_{\mu=1}^{n-k}$ of $\omega_i^{(k)}$ and the zeros $\{\gamma_{j, \mu}\}_{\mu=1}^{n-k}$ of $\omega_j^{(k)}$ interlace strictly. Furthermore, since the zeros of $\omega_i$ are less than or equal to the corresponding zeros of $\omega_j$, we have the following arrangement:

$$-1 < \gamma_{0, n-k} < \ldots < \gamma_{n, n-k} < \gamma_{0, n-k-1} < \ldots < \gamma_{n, n-k-1} < \ldots < \gamma_{0, 1} < \ldots < \gamma_{n, 1} < 1.$$
Since \( \omega_{j-1}(t_{j-1})/\omega_j(t_j) < 0 \) for \( j = 1, \ldots, n \), the above inequalities show that for 
\( x \in [-1, 1] \setminus \{ \gamma_{i,j} \}_{i=0,j=1}^{n-k} \), the quantities \( \{ \omega_j^{(k)}(x)/\omega_j(t_j) \}_{j=0}^n \) either change their signs 
alternatively, if 
\[
 x \in I_{n,k}^0, \quad I_{n,k}^0 = I_{n,k}^0(\Delta) := [-1, \gamma_{0,n-k}) \cup \bigcup_{j=n-k}^1 (\gamma_{n,j}, \gamma_{0,j-1}) \cup (\gamma_{n,1}, 1),
\]
or change signs alternatively with only one exception \( \frac{\omega_{\nu-1}^{(k)}(x)}{\omega_{\nu-1}(t_{\nu-1})} \frac{\omega_{\nu}^{(k)}(x)}{\omega_{\nu}(t_{\nu})} > 0 \) for some 
\( \nu \in \{ 1, \ldots, n \} \). The latter situation occurs when \( x \in I_{n,k}^\nu \), where 
\[
 I_{n,k}^\nu = I_{n,k}^\nu(\Delta) := \bigcup_{j=1}^{\nu-k} (\gamma_{\nu-1,j}, \gamma_{\nu,j}).
\]
Correspondingly, if \( x \in I_{n,k}^\nu \) for some \( \nu \in \{ 0, \ldots, n \} \), then (4) holds with equality sign 
for \( f = P_\nu \). If \( x = \gamma_{\nu,j} \), then \( \omega_{\nu}^{(k)}(x) = 0 \), and equality in (4) holds for \( f = P_\nu \) as well 
as for any \( x \in I_{n,k}^\nu \), which coincides with \( P_\nu \) at the points \( \{ t_j : j \neq \nu \} \).

Thus, in (4) equality holds for \( f = P_\nu \), if \( x \in I_{n,k}^\nu(\Delta) \), where 
\( \nu \in \{ 0, \ldots, n \} \) and since 
\[
 \bigcup_{\nu=0}^n I_{n,k}^\nu = [-1, 1],
\]
the proof of Lemma 2 is completed.

**Remark 1.** It follows from the proof of Lemma 2 that if for some \( f \in \Omega_n(\Delta, \epsilon) \) we 
have equality in (4) for some \( x \in I_{n,k}^\nu \) \( (\nu \in \{ 0, \ldots, n \}) \), then necessarily \( f = cP_\nu \), 
where \( c \) is a constant with \( |c| = 1 \). Thus, for \( x \in [-1, 1] \setminus \{ \gamma_{i,j} \}_{i=0,j=1}^{n-k} \), any extremal 
polynomial in Lemma 2 is of the form \( f = cP_\nu \), where \( \nu \in \{ 0, \ldots, n \} \) and \( |c| = 1 \).

**Proof of Theorem 2.** Set \( \epsilon_j := |Q(t_j)|, \quad j = 0, \ldots, n \), and define polynomials 
\( \{ P_\nu \}_{\nu=0}^n \) as above. Based on the interlacing assumption, we conclude that \( P_0 = Q \) or 
\( P_0 = -Q \), while for \( \nu = 1, \ldots, n \) the sign patterns of \( P_\nu \) and \( Q_\nu \) coincide. Moreover, we have 
\[
 |Q_\nu(t_j)| = \epsilon_j \frac{1 - x_j t_j}{|f_j - x_\nu|} \geq \epsilon_j \text{ for } j = 0, \ldots, n \text{ and } \nu = 1, \ldots, n.
\]

In the proof of Lemma 2, we deduced that for any \( f \in \Omega_n(\Delta, \epsilon) \) 
\[
 |f^{(k)}(x)| \leq |P_\nu^{(k)}(x)| \text{ if } x \in I_{n,k}^\nu, \quad \nu = 0, \ldots, n.
\]

For \( \nu = 0 \) (5) reads \( |f^{(k)}(x)| \leq |Q^{(k)}(x)| \), while for \( x \in I_{n,k}^\nu(\nu \in \{ 1, \ldots, n \}) \) we have 
\[
 |P_\nu^{(k)}(x)| = \sum_{j=0}^n \left| \frac{\omega_j^{(k)}(x)}{\omega_j(t_j)} \right| \epsilon_j \leq \sum_{j=0}^n \left| \frac{\omega_j^{(k)}(x)}{\omega_j(t_j)} \right| |Q(t_j)| = |Q_\nu^{(k)}(x)|
\]
(for the last equality we used that \( P_\nu \) and \( Q_\nu \) have the same sign pattern). The claim 
of Theorem 2 now follows from Lemma 2.

As an immediate consequence of Theorem 2 we get

**Corollary 1.** If, in addition to the assumptions of Theorem 2, for an \( k \in \{ 1, \ldots, n \} \) 
\[
 \max_{1 \leq \nu \leq n} \| Q_\nu^{(k)} \| \leq \| Q^{(k)} \|,
\]

then 
\[
 \| f^{(k)} \| \leq \| Q^{(k)} \|.
\]
3 Proof of Theorem 1

The proof of Theorem 1 follows from Corollary 1, applied to $Q = T_n$ and $k = 1$. The application of Corollary 1 is possible because of the following lemma:

Lemma 3. Let the polynomials $\{P_\nu\}_{\nu=1}^n$ be defined by

$$P_\nu(x) := T_n(x) \frac{1 - \xi_\nu x}{x - \xi_\nu}. $$

Then, for $n \geq 2$,

$$\|P'_\nu\| < n^2 \quad (\nu = 1, \ldots, n). \quad (6)$$

For $n = 2, 3$ the validity of (6) is verified directly, therefore we assume in what follows $n \geq 4$. The proof of Lemma 3 goes through a number of lemmas.

Lemma 4. For every $x \in [-1, 1]$ and for $\nu = 1, \ldots, n$

$$|P'_\nu(x)| \leq R_\nu(x),$$

where

$$R_\nu(x) = \left[\frac{(1 - \xi_\nu^2)^2}{(x - \xi_\nu)^4} + \frac{n^2(1 - \xi_\nu x)^2}{(1 - x^2)(x - \xi_\nu)^2}\right]^{1/2}$$

Proof. The result is immediate from

$$P'_\nu(x) = T'_n(x) \frac{1 - \xi_\nu x}{x - \xi_\nu} - T_n(x) \frac{1 - \xi_\nu^2}{(x - \xi_\nu)^2}, \quad (7)$$

the identity $|T_n(x)|^2 + (1 - x^2)|T'_n(x)|^2/n^2 = 1$, and Cauchy’s inequality. \qed

Lemma 5. $R_\nu(x)$ is a strictly convex function on each of the intervals $(-1, \xi_\nu)$ and $(\xi_\nu, 1)$.

Proof. We suppress the index $\nu$, writing

$$R(x) = \left[\frac{(1 - \xi^2)^2}{(x - \xi)^4} + \frac{n^2(1 - \xi x)^2}{(1 - x^2)(x - \xi)^2}\right]^{1/2} =: (g_1^2(x) + g_2^2(x))^{1/2},$$

where

$$g_1(x) := \frac{1 - \xi^2}{(x - \xi)^2}, \quad g_2(x) := \frac{n(1 - \xi x)}{(1 - x^2)^{1/2}(x - \xi)}. $$

Since

$$R'' = \frac{(g_1 g_2' - g_1' g_2)^2 + R^2(g_1'' g_2 + g_2'' g_1)}{R^3},$$
the lemma will be proved if we show that \( g_1(x)g_2''(x) \) and \( g_2(x)g_2''(x) \) are positive in \((-1, \xi)\) and in \((\xi, 1)\). This is easily seen for the first term, while for the second term a short calculation yields

\[
\frac{(x-\xi)^4(1-x^2)^3}{n^2}g_2(x)g_2''(x) = 2(1-\xi^2)(1-x^2)^2 - 2x(x-\xi)(1-\xi^2)(1-x^2) + (1-\xi x)(x-\xi)^2(2x^2 + 1).
\]

The positivity of the right hand side is easily verified with the help of the inequality

\[
2(1-\xi^2)(1-x^2)^2 + (1-\xi x)(x-\xi)^2(2x^2 + 1) \geq 2(1-x^2)|x-\xi||2(1-\xi^2)(1-\xi x)(2x^2 + 1)|^{1/2}.
\]

We now examine the polynomials \( \{P_\nu\}_{\nu=1}^n \). Due to symmetry, we may (and shall) consider only half of them, say, those with indices \( 1 \leq \nu \leq [(n + 1)/2] \). Recall that the zeros of \( P_\nu \) coincide with the zeros \( \{\xi_j\}_{j=1}^n \) of \( T_n \) with the exception of \( \xi_\nu \) which is replaced by \( 1/\xi_\nu \) (in the case \( n \) odd and \( \nu = (n + 1)/2 \), 1/\( \xi_\nu \) is interpreted as a zero at \( \infty \)). With this last convention, we observe that for \( 1 \leq \nu \leq [(n + 1)/2] \) the zeros of \( P_\nu \) are located to the right with respect to the zeros \( \{\xi_j\} \) of \( T_n \), and interlace with them. In view of Lemma 1, the same relation holds between the zeros of the derivatives of \( P_\nu \) and \( T_n \). We are interested in the behavior of \( P_\nu'(x) \), in particular, its critical points. To this end, we shall exploit (7) and the explicit form of \( P_\nu'' \),

\[
P_\nu''(x) = T_n''(x) \frac{1 - \xi_\nu x}{x - \xi_\nu} - 2T_n'(x) \frac{1 - \xi_\nu^2}{(x - \xi_\nu)^2} + 2T_n(x) \frac{1 - \xi_\nu^2}{(x - \xi_\nu)^3}.
\]  

In the proof of the next lemmas we shall use the differential equation

\[
(1 - x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0,
\]

as well as the following simple facts:

\[
\{n \sin(\alpha \pi/n)\}_{n=1}^\infty \not\to \alpha \pi,
\]

\[
\cot \alpha \leq \frac{1}{\alpha},
\]

where \( 0 < \alpha \leq \pi/2 \).

**Lemma 6.** The polynomials \( P_\nu \) \( (\nu = 1, \ldots, [(n + 1)/2]) \) satisfy the following:

(i) If \( 2 \leq \nu < \frac{n+1}{2} \), then \( P_\nu \) has exactly one local extremum to the right of 1;

(ii) \( P_\nu \) has exactly one local extremum in \((\xi_{\nu+1}, \eta_{\nu})\);

(iii) \( P_\nu \) is strictly monotone in \([\eta_{\nu}, \eta_{\nu-1}]\);

(iv) \( P_\nu \) is strictly monotone in \([-1, \eta_{n-1}] \) and in \([\eta_1, 1]\).
Proof. The first claim in (iv) follows trivially, since, as was already mentioned, the zeros of \( P_\nu \) are located to the right with respect to \( \{ \xi_j \}_{j=1}^n \). In view of Lemma 1, the same is true for the zeros of \( P''_\nu \) and \( T''_n \). Since the leftmost zero of \( T''_n \) is located to the right of \( \eta_{n-1} \), so is the smallest zero of \( P''_\nu \).

Substituting \( x = 1 \) in (8) we get

\[
P''_\nu(1) = \frac{n^2(n^2 - 1)}{3} - 2n^2 \cot^2 \left( \frac{(2\nu - 1)\pi}{4n} \right) + \frac{\cot^2 \left( \frac{(2\nu - 1)\pi}{4n} \right)}{4n}.
\]

With the help of (10) and (for \( \nu = 2 \) (11), it is easy to see that \( P''_\nu(1) > 0 \) for \( 2 \leq \nu \leq \left[ \frac{n+1}{2} \right] \). Since \( P'_\nu \) has a negative leading coefficient and at most one critical point to the right of \( x = 1 \), this proves part (i) of the lemma.

Now we find the sign of \( P''_\nu \) at the points \( \xi_{\nu+1} \), \( \eta_{\nu} \), and \( \eta_{\nu-1} \). First, we shall show that

\[
\text{sign} \{ P''_\nu(\xi_{\nu+1}) \} = (-1)^{\nu+1}.
\]  
(12)

Putting \( x = \xi_{\nu+1} \) in (8) and using that \( T''_n(\xi_{\nu+1}) = \xi_{\nu+1}T'_n(\xi_{\nu+1})/(1 - \xi_{\nu+1})^2 \) and \( \text{sign} \{ T^n_n(\xi_{\nu+1}) \} = (-1)^{\nu} \), we get

\[
\text{sign} \{ P''_\nu(\xi_{\nu+1}) \} = (-1)^{\nu+1}\text{sign} \left\{ 2(1 - \xi_{\nu}^2)(1 - \xi_{\nu}^2)(1 - \xi_{\nu+1}) + \xi_{\nu+1}(\xi_{\nu} - \xi_{\nu+1})(1 - \xi_{\nu}\xi_{\nu+1}) \right\}.
\]

Now (12) is obvious if \( \xi_{\nu+1} > 0 \). The only possibility where \( \xi_{\nu+1} < 0 \) is \( \nu = m \) and \( n = 2m \) or \( n = 2m - 1 \). An easy calculation shows that for \( n \geq 4 \) (12) is true in this case, too.

Next, we prove both (ii) and (iii) by showing that

\[
\text{sign} \{ P''_\nu(\eta_{\mu}) \} = (-1)^\nu \text{ for } \mu = \nu, \nu - 1, \mu \neq 0.
\]  
(13)

Using (8) and (9), we obtain

\[
P''_\nu(\eta_{\mu}) = \frac{T'_n(\eta_{\mu})}{(\xi_{\nu} - \eta_{\mu})^3(1 - \eta_{\mu}^2)} \left[ n^2(1 - \xi_{\nu}\eta_{\mu})(\xi_{\nu} - \eta_{\mu})^2 - 2(1 - \xi_{\nu}^2)(1 - \eta_{\mu}^2) \right].
\]  
(14)

Since \( \text{sign} \{ T'_n(\eta_{\mu}) \} = (-1)^\nu \), it suffice to prove that the term in the square brackets is positive. Using the inequality \( (1 - \xi_{\nu}^2)(1 - \eta_{\mu}^2) < (1 - \xi_{\nu}\eta_{\mu})^2 \) we obtain

\[
n^2(1 - \xi_{\nu}\eta_{\mu})(\xi_{\nu} - \eta_{\mu})^2 - 2(1 - \xi_{\nu}^2)(1 - \eta_{\mu}^2) > (1 - \xi_{\nu}\eta_{\mu})^2 - 2(1 - \xi_{\nu}\eta_{\mu})^2.
\]

After simple manipulations, using the trigonometric representation of \( \xi_{\nu} \) and \( \eta_{\mu} \) we find that the inequality \( n^2(\xi_{\nu} - \eta_{\mu})^2 - 2(1 - \xi_{\nu}\eta_{\mu}) \geq 0 \) is equivalent to

\[
\frac{1}{n^2\sin^2 \frac{\pi}{4n}} + \frac{1}{n^2\sin^2 \frac{2\nu - 1\pi}{4n}} \leq 2.
\]

This last inequality will hold for all \( \nu \in \{1, \ldots, \left[ \frac{n+1}{2} \right] \} \) and \( \mu = \nu, \nu - 1, (\mu \neq 0) \), if it is true for \( \nu = \mu = 1 \), i.e., if

\[
\frac{1}{n^2\sin^2 \frac{\pi}{4n}} + \frac{1}{n^2\sin^2 \frac{2\pi}{4n}} \leq 2.
\]
Since the left hand side is a decreasing function of $n$ (see (10)), and for $n = 3$ it is $(\sin^{-2}(\pi/12) + 2)/9 = (4\sqrt{3} + 10)/9 < 2$, (13) is proved. Now we conclude from (12) and (13) (with $\mu = \nu$) that $P''_{\nu}$ has a zero in $(\xi_{\nu + 1}, \eta_{\nu})$ for $\nu = 1, \ldots, [(n + 1)/2]$. In addition, (13) implies that this zero is unique, and no zeros of $P''_{\nu}$ exist in $[\eta_{\nu - 1}, \xi_{\nu - 1}]$ ($\nu \geq 2$), otherwise there would be at least three zeros in $(\xi_{\nu + 1}, \xi_{\nu - 1})$, a contradiction. For the same reason, $P''_{1}$ has a simple zero in $(\xi_{2}, \eta_{1})$, and no zeros of $P''_{1}$ exist in $[\eta_{1}, 1]$. This is exactly the claim of (iii) for $\nu = 1$ and of the second part of (iv) for $\nu = 1$.

To prove the second part of (iv) for $2 \leq \nu < (n + 1)/2$, we shall show that

$$P''_{\nu}(\eta_{1}) > 0. \quad (15)$$

Having established (15), the second part of (iv) will follow immediately. Indeed, we found in the beginning of this proof that $P''_{\nu}(1) > 0$ for $2 \leq \nu < (n + 1)/2$, and if $P'_{\nu}$ was not monotone in $[\eta_{1}, 1]$, then $P''_{\nu}$ would have at least three zeros (two zeros, if $\nu = (n + 1)/2$) to the right of $\eta_{1}$, which is impossible. The proof of (15) goes along the lines of the proof of (13). Equation (14) with $\mu = 1$ shows that (15) follows if

$$n^{2}(1 - \xi_{\nu}\eta_{1})(\xi_{\nu} - \eta_{1})^{2} - 2(1 - \xi_{\nu}^{2})(1 - \eta_{1}^{2}) > 0,$$

or, in view of $(1 - \xi_{\nu}^{2})(1 - \eta_{1}^{2}) \leq (1 - \xi_{\nu}\eta_{1})^{2}$, if

$$n^{2}(\xi_{\nu} - \eta_{1})^{2} - 2(1 - \xi_{\nu}\eta_{1}) > 0.$$

The latter inequality is equivalent to the inequality

$$\frac{1}{n^{2}\sin^{2}\frac{(2\nu-3)\pi}{4n}} + \frac{1}{n^{2}\sin^{2}\frac{(2\nu+1)\pi}{4n}} \leq 2,$$

whose validity is easily verified with the help of (10). Lemma 6 is proved. ∎

**Lemma 7.** The following estimates for $\|P'_{\nu}\|$ hold true:

(i) For $\nu = 1, 2,$

$$\|P'_{\nu}\| \leq \max\{|P'_{\nu}(-1)|, |P'_{\nu}(1)|, R_{\nu}(\eta_{n-1}), R_{\nu}(\eta_{1})\};$$

(ii) For $\nu = 3, \ldots, [(n + 1)/2],

$$\|P'_{\nu}\| \leq \max\{|P'_{\nu}(-1)|, |P'_{\nu}(1)|, R_{\nu}(\eta_{n-1}), R_{\nu}(\eta_{\nu}), R_{\nu}(\eta_{\nu-1}), R_{\nu}(\eta_{1})\}.$$

**Proof.** According to Lemma 6, $P'_{1}$ is monotone in $[-1, \eta_{n-1}]$ and $[\eta_{1}, 1]$, therefore on these intervals

$$|P'_{1}(x)| \leq \max\{|P'_{1}(-1)|, |P'_{1}(\eta_{n-1})|, |P'_{1}(\eta_{1})|, |P'_{1}(1)|\}.$$
On the complementary interval $[\eta_{n-1}, \eta_1]$, we have $|P'_1(x)| \leq R_1(x)$ (Lemma 4), and since $R_1$ is convex there (Lemma 5), it follows that $R_1(x) \leq \max\{R_1(\eta_{n-1}), R_1(\eta_1)\}$ for $x \in [\eta_{n-1}, \eta_1]$. This proves (i) for $\nu = 1$.

The proof of (i) for $\nu = 2$ relies on the observation that, by Lemma 6, $P'_2$ is monotone in $[-1, \eta_{n-1}]$ and $[\eta_2, 1]$, while $|P'_2(x)| \leq \max\{R_2(\eta_{n-1}), R_2(\eta_2)\}$ in $[\eta_{n-1}, \eta_2]$, by virtue of Lemmas 4 and 5.

Part (ii) can be proved in the same fashion, exploiting the monotonicity of $P'_\nu$ on the intervals $[-1, \eta_{n-1}]$, $[\eta_\nu, \eta_{\nu-1}]$ and $[\eta_1, 1]$, and the convexity of $R_\nu$ on $[\eta_{n-1}, \eta_\nu]$ and $[\eta_{\nu-1}, \eta_1]$. We leave the details to the reader.

Our last lemma estimates the quantities appearing in Lemma 7.

**Lemma 8.** The following inequalities hold true:

(i) $|P'_\nu(\pm 1)| < n^2 (\nu = 1, \ldots, [(n+1)/2]);$
(ii) $R_\nu(\eta_1) < n^2 (\nu = 1, 3, 4, \ldots, [(n+1)/2]);$
(iii) $R_\nu(\eta_\nu) < n^2 (\nu = 1, \ldots, [(n+1)/2]);$
(iv) $R_\nu(\eta_{\nu-1}) < n^2 (\nu = 3, \ldots, [(n+1)/2]);$
(v) $R_\nu(\eta_{n-1}) < n^2 (\nu = 1, \ldots, [(n+1)/2]).$

**Proof.** Substituting $x = \pm 1$ in (7) we get

$$P'_\nu(1) = n^2 - \cot^2 \frac{(2\nu - 1)\pi}{4n}, \quad |P'_\nu(-1)| = n^2 - \tan^2 \frac{(2\nu - 1)\pi}{4n}.$$  

Then (10) and $0 < (2\nu - 1)\pi/(2n) \leq \pi/4$ show the validity of a slightly sharper inequalities than (i), namely

$$n^2 - 1 \leq |P'_\nu(-1)| < n^2 - \frac{\pi}{4n}$$

and

$$16/\pi^2 n^2 < P'_\nu(1) < n^2 - 1.$$  

Now, we prove (ii). A short calculation yields

$$R_\nu(\eta_1) = \left[ \frac{(1 - \xi_\nu^2)^2}{(\eta_1 - \xi_\nu)^4} + \frac{n^2(1 - \xi_\nu \eta_1)^2}{(1 - \eta_1^2)(\eta_1 - \xi_\nu)^2} \right]^{1/2} = \{[A(\nu)]^2 + [B(\nu)]^4\}^{1/2}, \tag{16}$$

where

$$A(\nu) = \frac{n}{2} |2 \cot \frac{\pi}{n} + \cot \frac{(2\nu - 3)\pi}{4n} - \cot \frac{(2\nu + 1)\pi}{4n}|,$$

$$B(\nu) = \frac{1}{2} |\cot \frac{(2\nu - 3)\pi}{4n} + \cot \frac{(2\nu + 1)\pi}{4n}|.$$
Assume first that $3 \leq \nu \leq [(n + 1)/2]$, then it is easy to see that $A(\nu) \leq A(3)$ and $B(\nu) \leq B(3)$. We use (11) to obtain

$$
B(3) = \frac{1}{2} \left[ \cot \frac{3\pi}{4n} + \cot \frac{7\pi}{4n} \right] < \frac{20n}{21\pi},
$$

$$
A(3) = \frac{n}{2} \left[ \cot \frac{3\pi}{4n} + 2 \cot \frac{\pi}{n} - \cot \frac{7\pi}{4n} \right] < \frac{n}{2} \left[ \cot \frac{3\pi}{4n} + 2 \cot \frac{\pi}{n} \right] < \frac{5n^2}{3\pi}.
$$

Therefore, for $3 \leq \nu \leq [(n + 1)/2],

$$
R_\nu(\eta_1) < \left[ \left( \frac{5n^2}{3\pi} \right)^2 + \left( \frac{20n}{21\pi} \right)^4 \right]^{1/2} < 0.54n^2 < n^2.
$$

Similarly, for $\nu = 1$, we find

$$
A(1) = \frac{n}{2} \left[ \cot \frac{\pi}{4n} - 2 \cot \frac{\pi}{n} + \cot \frac{3\pi}{4n} \right] < \frac{n}{2} \left[ \cot \frac{\pi}{4n} + \cot \frac{3\pi}{4n} \right] < \frac{8n^2}{3\pi},
$$

$$
B(1) = \frac{1}{2} \left[ \cot \frac{\pi}{4n} - \cot \frac{3\pi}{4n} \right] < \frac{1}{2} \cot \frac{\pi}{4n} < \frac{2n}{\pi}.
$$

Hence,

$$
R_1(\eta_1) < \left[ \left( \frac{8n^2}{3\pi} \right)^2 + \left( \frac{2n}{\pi} \right)^4 \right]^{1/2} < 0.95n^2 < n^2.
$$

Thus, (ii) is proved.

Next, we prove (iii). For $1 \leq \nu \leq [(n + 1)/2]$, we have

$$
R_\nu(\eta_\nu) = \left[ \frac{(1 - \xi_\nu^2)}{(\xi_\nu - \eta_\nu)^4} + \frac{n^2(1 - \xi_\nu^2)}{(1 - \eta_\nu^2)(\xi_\nu - \eta_\nu)^2} \right]^{1/2} = \{[C(\nu)]^2 + [D(\nu)]^4\}^{1/2},
$$

where

$$
C(\nu) = \frac{n}{2} \left[ \cot \frac{\pi}{4n} + \cot \frac{(4\nu - 1)\pi}{4n} - 2 \cot \frac{\nu\pi}{n} \right],
$$

$$
D(\nu) = \frac{1}{2} \left[ \cot \frac{\pi}{4n} - \cot \frac{(4\nu - 1)\pi}{4n} \right].
$$

Unlike the situation with $A(\nu)$ and $B(\nu)$, we observe that $C(\nu)$ and $D(\nu)$ increase with $\nu$, and for $n \geq 3$

$$
D(\nu) \leq D([(n + 1)/2]) = \frac{n}{n \sin \frac{\pi}{2n}} \leq \frac{2n}{3},
$$
\[
C(\nu) \leq C((n+1)/2) = \frac{n}{2} \left[ \cot \frac{\pi}{4n} + 2 \tan \frac{\pi}{2n} - \tan \frac{\pi}{4n} \right] \\
= \frac{n}{\sin \frac{\pi}{2n}} + n \left[ \tan \frac{\pi}{2n} - \tan \frac{\pi}{4n} \right] \\
< \frac{n^2}{n \sin \frac{\pi}{2n}} + \frac{\pi}{4 \cos^2 \frac{\pi}{2n}} \\
\leq \frac{1}{3}(2n^2 + \pi).
\]

With this (iii) is proved, since

\[
R_\nu(\eta_\nu) < n^2 \left[ \left( \frac{2}{3} + \frac{\pi}{3n^2} \right)^2 + \left( \frac{2}{3} \right)^4 \right]^{1/2} < 0.91n^2 < n^2.
\]

The same arguments as above lead to the proof of (iv): \( R_\nu(\eta_{\nu-1}) = [(\bar{C}(\nu))^2 + (\bar{D}(\nu))^4]^{1/2} \), where

\[
\bar{C}(\nu) = \frac{n}{2} \left[ \cot \frac{\pi}{4n} + 2 \cot \frac{(n+1)\pi}{n} - \cot \frac{(4\nu - 3)\pi}{4n} \right],
\]

\[
\bar{D}(\nu) = \frac{1}{2} \left[ \cot \frac{\pi}{4n} + \cot \frac{(4\nu - 3)\pi}{4n} \right].
\]

Observing that \( \bar{C}(\nu) \) and \( \bar{D}(\nu) \) decrease with \( \nu \), for \( 3 \leq \nu \leq \lfloor (n+1)/2 \rfloor \) we find the estimates

\[
\bar{D}(\nu) \leq \bar{D}(3) = \frac{1}{2} \left[ \cot \frac{\pi}{4n} + \cot \frac{9\pi}{4n} \right] < \frac{20n}{9\pi},
\]

\[
\bar{C}(\nu) \leq \bar{C}(3) = \frac{n}{2} \left[ \cot \frac{\pi}{4n} + 2 \cot \frac{2\pi}{n} - \cot \frac{9\pi}{4n} \right] \\
< \frac{n}{2} \left[ \cot \frac{\pi}{4n} + \cot \frac{7\pi}{4n} \right] \\
< \frac{16n^2}{7\pi},
\]

and hence

\[
R_\nu(\eta_{\nu-1}) < \left[ \left( \frac{16n^2}{7\pi} \right)^2 + \left( \frac{20n}{9\pi} \right)^4 \right]^{1/2} < 0.89n^2 < n^2.
\]

Finally, (v) can be proved in the same way as (i)–(iv). Alternatively, one can use the inequality

\[
\frac{1 - \xi \eta}{|\xi - \eta|} \geq \frac{1 + \xi \eta}{\xi + \eta} \quad (0 \leq \xi, \eta < 1, \xi \neq \eta)
\]

to compare pairwise \( A(\nu) \) and \( B(\nu) \) with the corresponding terms appearing in \( R_\nu(\eta_{\nu-1}) = R_\nu(-\eta_1) \). The result is \( R_\nu(\eta_{\nu-1}) \leq R_\nu(\eta_1) < n^2 \). We omit the details.  

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Proof of Lemma 3. The inequality follows from Lemmas 7 and 8.  

Proof of Theorem 1. Inequality (3) follows immediately from Corollary 1 and Lemma 3. It remains to clarify in which cases a equality is possible. Let \( \Delta = \{ t_j \}_{j=0}^n \) be a fixed mesh satisfying the assumptions of Theorem 1. Let \( \epsilon = (\epsilon_0, \ldots, \epsilon_n) =: ([T_n(t_0)], \ldots, [T_n(t_n)]) \), and the polynomials \( P_0 = T_n, \ P_\nu (\nu = 1, \ldots, n) \) be defined as in Section 2. Suppose that \( f \in \Omega(\Delta, \epsilon) \) is an extremal polynomial, i.e., \( \| f' \| = n^2 \). According to Remark 1 and Lemma 3, for \( x \in \cup_{\nu=1}^n T'_{n,1} \) there holds

\[
|f'(x)| \leq \max_{1 \leq \nu \leq n} \| P'_\nu \| < n^2,
\]

therefore \( \| f' \| \) is attained for \( x \in I_{n,1}^0 \). However, when \( x \in I_{n,1}^0 \) we have

\[
|f'(x)| \leq |P_0'(x)| = |T'_0(x)| \leq T'_n(1) = n^2,
\]

and equality holds only for \( x = \pm 1 \) and \( f = cT_n \) with \( |c| = 1 \). Theorem 1 is proved.  

4 Concluding remarks

1. The requirement in Theorem 1 that the points \( \Delta = \{ t_j \}_{j=0}^n \) interlace strictly with the zeros of \( T_n \) was only imposed in order to avoid unimportant complications in the proof. Actually, Theorem 1 is valid under the weaker assumption that \( \{ t_j \}_{j=0}^n \) interlace with \( \{ \xi_j \}_{j=1}^n \). If a comparison point \( t_j \) coincides with a zero of \( T_n \), then the polynomials from the corresponding class \( \Omega_n(\Delta, \epsilon) \) must vanish at that point. In the case when all \( \{ \xi_j \}_{j=1}^n \) belong to \( \Delta \) Theorem 1 holds trivially, since in that case \( \Omega_n(\Delta, \epsilon) = \{ cT_n(x) : |c| \leq 1 \} \).

2. So far, we cannot extend Theorem 1 to higher order derivatives, i.e., to prove \( \| f^{(k)} \| \leq \| T^{(k)}_n \| \) for all \( k \geq 2 \). However, it should be pointed out that this inequality holds true for \( k = n - 1 \) and for \( k = n \). This is easily seen from the proof of Lemma 2: for any polynomial \( f \in \Omega_n(\Delta, \epsilon) \) and for \( k = n - 1, n \) we have \( \| f^{(k)} \| = |f^{(k)}(-1)| \) or \( \| f^{(k)} \| = |f^{(k)}(1)| \), and for \( x = \pm 1 \) the extremal polynomials in Lemma 2 are of the form \( cP_0 = \pm cT_n, |c| = 1 \).

3. According to Lemma 2, a necessary condition for a mesh \( \Delta = \{ t_j \}_{j=0}^n \) to admit DS-inequality with an extremal polynomial \( Q = T_n \) is, the sign pattern of \( (T_n(t_0), \ldots, T_n(t_n)) \) to coincide (up to a factor -1) with the sign pattern of some of the polynomials \( \{ P_\nu \}_{\nu=0}^n \). Theorem 1 asserts DS-inequality for all meshes \( \Delta \) having the sign structure of \( P_0 \). One may think that DS-inequality also holds for any other mesh \( \Delta = \{ t_j \}_{j=0}^n \) for which the sign pattern of \( (T_n(t_0), \ldots, T_n(t_n)) \) coincides with the sign pattern of some \( P_\nu, \nu \in \{ 1, \ldots, n \} \). However, the example below shows that this is not true, in general.
Let $t_j = \eta_{j+1}$ for $j = 0, 1, \ldots, n-2$, $t_n = \eta_n$ and $t_{n-1} = \zeta$, where $\zeta \in (-1, \xi_n)$. Define polynomial

$$q(x) = \begin{cases} 
T_n(x) & \text{for } x = t_j, \quad j = 0, \ldots, n-2, n, \\
-T_n(x) & \text{for } x = t_{n-1}.
\end{cases}$$

Clearly, $q$ has the same sign structure as $P_{n-1}$, and $|q(t_j)| = |T_n(t_j)|$ ($j = 0, \ldots, n$). The explicit form of $q$ is

$$q(x) = T_n(x) + a(1 + x)T'_n(x),$$

where $a = -2T_n(\zeta)/(1 + \zeta)(1 + \zeta) > 0$,

and for $k = 1, \ldots, n$ we have

$$\|q^{(k)}\| \geq q^{(k)}(1) > T_n^{(k)}(1) = \|T_n^{(k)}\|.$$ 

4. As was mentioned in [8, p. 174], inequalities of DS-type may be viewed as exact estimates for the roundoff error in Lagrange differentiation formulas. We describe briefly a possible application of the result of Theorem 1.

Let $\Delta = \{t_j\}_{j=0}^n$ be a mesh whose points interlace strictly with the zeros of $T_n$. Suppose that inaccurate data $\{\tilde{f}(t_j)\}_{j=0}^n$ for a function $f \in C^{n+1}[-1, 1]$ is given, where

$$|f(t_j) - \tilde{f}(t_j)| \leq \delta_j$$

($j = 0, \ldots, n$).

If $f'(x) \approx L'_n(\tilde{f}; x)$ is the Lagrange differentiation formula based on this information, then for the error $R(f; x) := f'(x) - L'_n(\tilde{f}; x)$ there holds

$$R(f; x) = R^{\text{round}}(f; x) + R^{\text{trunc}}(f; x)$$

with $R^{\text{round}}(f; x) = L'_n(\tilde{f} - f; x)$ being the error caused by inaccuracy of the data and $R^{\text{trunc}}(f; x)$ the error caused by the fact that $f$ is not necessarily a polynomial (truncation error). We have the estimate

$$\|R(f; \cdot)\| \leq \|R^{\text{round}}(f; \cdot)\| + \|R^{\text{trunc}}(f; \cdot)\|.$$ 

The exact bound for the truncation error in the Lagrange differentiation formula in the general case has been obtained by Shadrin [14] (in our case $\|R^{\text{trunc}}(f; \cdot)\| \leq \|f^{(n+1)}(\omega')/(n+1)! \|$. For the roundoff error, Theorem 1 provides the following exact upper bound:

$$\|R^{\text{round}}(f; \cdot)\| \leq Mn^2, \quad \text{where} \quad M = \max_{0 \leq j \leq n} \frac{\delta_j}{|T_n(t_j)|}.$$ 

This upper bound is attained when $\delta_j/|T_n(t_j)| = M$ for $j = 0, \ldots, n$. 

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References


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