Industrial Mathematics Institute

2004:13

A singularly perturbed convection-diffusion problem in a half-plane

R.B. Kellogg and M. Stynes

IMI
Preprint Series

Department of Mathematics
University of South Carolina

A singularly perturbed convection-diffusion problem in a half-plane*

R. Bruce Kellogg† and Martin Stynes‡

August 13, 2004

Abstract

A singularly perturbed convection-diffusion equation with constant coefficients is considered in a half plane, with Dirichlet boundary conditions. The boundary function has a specified degree of regularity except for a jump discontinuity, or jump discontinuity in a derivative of specified order, at a point. Precise pointwise bounds for the derivatives of the solution are obtained. The bounds show both the strength of the interior layer emanating from the point of discontinuity and the blowup of the derivatives resulting from the discontinuity, and make precise the dependence of the derivatives on the singular perturbation parameter.

1 Introduction

Singularly perturbed convection-diffusion problems (e.g., linearized Navier-Stokes equations at high Reynolds number) arise in many applications and the precise determination of the behavior of their solutions is of great interest. In particular, the numerical analysis of these problems has attracted the attention of many researchers during the last two decades—see [3, 7] and their bibliographies—and to carry out such analyses one needs to know a priori how the derivatives of the solution u of the problem depend on the singular perturbation parameter. Much research has gone into the provision of such estimates. For elliptic convection-diffusion problems on bounded domains, estimates of global Sobolev norms of u and some pointwise bounds are given in [2], while in [5] pointwise bounds are proved. Further related references are given in the discussion below.

The solution u to an elliptic convection-diffusion problem in a two-dimensional domain can in general exhibit boundary and interior layers. Thus the bounds on derivatives of u in [5] show that u has exponential boundary layers along two sides of the rectangular domain considered, certain bounds in [8] show a parabolic boundary layer in u, and the bounds in [6] show both exponential and parabolic boundary layers. Our earlier paper [4] extended the results of [6] by also considering the effects of incompatibilities in the problem data at corners of the domain.

Despite the wealth of information provided by the papers cited, none of them gives any information about the pointwise behavior of the solution u near interior layers. Such issues are examined in [9, Chapter IV], but many details are omitted and it is difficult to ascertain the precise assumptions made. The purpose of the present paper is to derive carefully pointwise bounds for the derivatives of the solution u of a simple singular perturbation problem whose solution contains an interior layer. We study a convection-diffusion problem on a half-plane. The interior layer is produced by a discontinuity in the boundary function, or a derivative of the boundary function, at a particular point of the boundary.

When solving a classical (non-singularly perturbed) elliptic boundary value problem in which the boundary function has a jump discontinuity, one obtains a solution whose derivatives become infinite near the point of discontinuity. The situation is illustrated by the function $\tan^{-1}(y/x)$, which is harmonic in the right half-plane but whose boundary values are $\pi/2$ for $y > 0$ and $-\pi/2$ for $y < 0$. If in addition the boundary value problem is singularly perturbed, to what extent does the pointwise behavior of its solution derivatives

*Research supported by the Department of Mathematics and the Industrial Mathematics Institute at the University of South Carolina, and by the Boole Centre for Research in Informatics at the National University of Ireland, Cork.

†Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA; kellogg@ipst.umd.edu

‡Department of Mathematics, National University of Ireland, Cork, Ireland; m.stynes@ucc.ie

depend on the singular perturbation parameter? We shall answer this question in the case of a singularly perturbed convection-diffusion equation in a half-plane.

We study the half-plane problem

$$Lu := -\varepsilon\Delta u + p_1 u_x + p_2 u_y + qu = f \quad \text{for } x > 0, \tag{1.1a}$$

$$u(0, y) = h(y) \quad \text{for } y \in \mathbb{R}. \tag{1.1b}$$

Here, p_1, q and ε are positive constants with $0 < \varepsilon \leq 1$. The coefficient p_2 is also constant but no assumption is made regarding its sign; it may even be zero. We denote by $\alpha = [p_1, p_2]$ the subcharacteristic direction and by $\beta = [-p_2, p_1]$ the direction normal to α. The function h is smooth on $[0, \infty)$ and has a smooth extension from $(-\infty, 0)$ to $(-\infty, 0]$, but is allowed to have a jump discontinuity at 0. An integer-valued parameter $\nu \geq -1$ is used to indicate the degree of discontinuity that h has at the origin: the value $\nu = -1$ means that $h(+0) \neq h(-0)$, while a value $\nu \geq 0$ means that

$$D^k h(+0) = D^k h(-0) \quad \text{for } k = 0, \cdots, \nu. \tag{1.2}$$

Our aim in this paper is to establish pointwise bounds for all derivatives of the solution u of (1.1) in terms of the smoothness of $h(y)$ for $y \neq 0$ and in terms of the degree ν to which h is discontinuous at $y = 0$. Furthermore, we shall make explicit the dependence of these bounds on the point of evaluation and on the small diffusion parameter ε. The solution of (1.1) will typically have an interior layer in its higher-order derivatives that lies along the subcharacteristic $p_1 y = p_2 x$ that passes through the origin in $\mathbb{R}^2$. Along such a layer the behaviour of the solution in the subcharacteristic direction (i.e., the direction of the characteristics of the reduced differential operator defined by $v \mapsto p_1 v_x + p_2 v_y + qv$) is expected to be very different from its behaviour in a direction perpendicular to those subcharacteristics. Thus we shall express our results using "subcharacteristic" derivatives defined by $D_\alpha = p_1 D_x + p_2 D_y$ and "cross-characteristic" derivatives defined by $D_\beta = -p_2 D_x + p_1 D_y$.

Notation. Throughout the analysis, c and C are used to denote generic constants that are independent of ε, u, x, y and ν. They can take different values in different places, even within the same argument. We also use the Sobolev Hilbert spaces $H^n(\mathbb{R})$ and $H^n(\mathbb{R}_\pm)$, $H^n(\mathbb{R}^2)$ and $H^n(\mathbb{R}^2_+)$, where we write $\mathbb{R}_+$ and $\mathbb{R}_-$ for the intervals $(0, \infty)$ and $(-\infty, 0)$ respectively, where $\mathbb{R}_\pm = \mathbb{R}_+ \cup \mathbb{R}_-$, and where $\mathbb{R}^2_+$ is the right half-plane $x > 0$.

To state the main result of the paper, let $r = \sqrt{x^2 + y^2}$ and let $d(x, y)$ denote the distance from a point (x, y) to the line $p_1 y = p_2 x$. That is, $d(x, y) = |p_2 x - p_1 y|/|\alpha|^2$ where $|\alpha| = \sqrt{p_1^2 + p_2^2}$.

Theorem 1.1. *Let m and n be non-negative integers. Let $h \in H^{2m+n+1}(\mathbb{R}_\pm)$, $f \in H^{m+n+2}(\mathbb{R}^2_+)$. Let ν be an integer with $-1 \leq \nu < n$ and suppose that either $\nu = -1$ or h satisfies (1.2). Then there is a constant C such that for $0 < \varepsilon \leq 1$ the solution u of (1.1) satisfies*

$$|D_\alpha^m D_\beta^n u(x, y)| \leq C(1 + r^{-m-n+\nu+1}) \text{ for } r \leq 2\varepsilon, \tag{1.3a}$$

$$|D_\alpha^m D_\beta^n u(x, y)| \leq C\left(1 + \varepsilon^{(-n+\nu+1)/2} r^{-m+(-n+\nu+1)/2} e^{-cd^2/\varepsilon} + r^{-m-n+\nu+1} e^{-cr/\varepsilon}\right) \text{ for } 2\varepsilon \leq r \leq 1. \tag{1.3b}$$

The bound (1.3a) shows that the solution $u(x, y)$ of (1.1) has the same singular behavior near the origin as the solution of the half-plane problem for the Laplace operator with a discontinuity in the $(\nu+1)$th derivative of the boundary data. (For the case $\nu = -1$, note the behavior of derivatives of the function $\tan^{-1}(y/x)$.) The bound (1.3b) shows a "parabolic" interior layer behavior of $u(x, y)$ on the subcharacteristic line $p_1 y = p_2 x$ when (x, y) is not close to the origin, and it shows how this interior layer changes as it approaches the singularity at the origin. Note that the bounds are equivalent when $r = 2\varepsilon$. In case the boundary function h has more regularity at the origin than is allowed by the theorem, derivative bounds for the solution are given in Lemma 2.1 below. Finally, we mention that derivative results related to the ones obtained here are obtained in [4] for the equation (1.1a) with $p_2 = 0$ and posed in the unit square, but interior layers were not considered in that paper.

2 Bounds on the convective derivatives

This section gives bounds on the convective derivatives of u. The analysis proceeds initially through an investigation of some special cases of the final result. Then these special cases are combined to obtain the desired bound.

The first special case is the situation where h has a certain amount of smoothness on all of $(-\infty, \infty)$. Then one expects the solution u to be well-behaved, in the sense that its lower-order derivatives should be bounded independently of ε, and this is borne out by the following result.

Lemma 2.1. *Let m and n be non-negative integers. Assume that $h \in H^{m+n+1}(\mathbb{R})$, $f \in H^{m+n+2}(\mathbb{R}^2_+)$. Then there exists a constant C such that the solution u of (1.1) satisfies*

$$|D_x^m D_y^n u(x,y)| \le C(\|h\|_{H^{m+n+1}(\mathbb{R})} + \|f\|_{H^{m+n+2}(\mathbb{R}^2_+)})$$

for $0 < \varepsilon \le 1$ and all (x, y) such that $x > 0$.

Proof. Let $F \in H^{m+n+2}(\mathbb{R}^2)$ be an extension of f to $\mathbb{R}^2$ such that $\|F\|_{H^{m+n+2}(\mathbb{R}^2)} \le C\|f\|_{H^{m+n+2}(\mathbb{R}^2_+)}$. Let U be the solution of the problem $LU = F$ in $\mathbb{R}^2$. From the assumption $q > 0$, an energy inequality can be applied to the problems $L(D_x^k D_y^\ell U) = D_x^k D_y^\ell F$, where $k + \ell = m + n + 2$, to obtain $\|U\|_{H^{m+n+2}(\mathbb{R}^2)} \le C\|F\|_{H^{m+n+2}(\mathbb{R}^2)}$. Using a Sobolev inequality and the boundedness of the extension it follows that $|D_x^m D_y^n U(x,y)| \le C\|f\|_{H^{m+n+2}(\mathbb{R}^2_+)}$. Again using one of the Sobolev inequalities, one has $\|U(0,\cdot)\|_{H^{m+n+1}(\mathbb{R})} \le C\|U\|_{H^{m+n+2}(\mathbb{R}^2_+)} \le C\|f\|_{H^{m+n+2}(\mathbb{R}^2_+)}$. Since $L(U - u) = 0$ on $\mathbb{R}^2_+$, it remains to prove the lemma in the case $f = 0$.

We use the Fourier transform

$$\hat{u}(x, \hat{y}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,y) e^{-y\hat{y}i} dy.$$

Denoting by $\hat{h}$ the Fourier transform of h, the problem (1.1) with $f \equiv 0$ transforms to

$$-\varepsilon \hat{u}_{xx} + p_1 \hat{u}_x + (\varepsilon \hat{y}^2 + q + p_1 \hat{y} i)\hat{u} = 0, \ \hat{u}(0, \hat{y}) = \hat{h}(\hat{y}). \tag{2.1}$$

To solve (2.1) write $\hat{u} = \hat{h}(\hat{y})e^{rx}$ where

$$\varepsilon r^2 - p_1 r - (\varepsilon \hat{y}^2 + q + p_2 \hat{y} i) = 0. \tag{2.2}$$

Let $\Delta = \sqrt{p_1^2 + 4\varepsilon^2 \hat{y}^2 + 4\varepsilon q + 4\varepsilon p_2 \hat{y} i}$. The square root is defined in the cut plane with the cut along the negative real axis, and producing positive values on the positive real axis. Since $\Re(p_1^2 + 4\varepsilon^2\hat{y}^2 + 4\varepsilon q + 4\varepsilon p_2 \hat{y} i) > 0$, one has $\arg \Delta^2 \in (-\pi/2, \pi/2)$, so $\arg \Delta \in (-\pi/4, \pi/4)$ and $\Re\Delta > 0$. We shall use the root

$$r = \frac{p_1 - \Delta}{2\varepsilon} = -\frac{2\varepsilon\hat{y}^2 + 2q + 2p_2\hat{y}i}{p_1 + \Delta}$$

of (2.2). Write $\Delta = \Delta_1 + \Delta_2 i$; then $\Delta_1^2 - \Delta_2^2 = \Re(\Delta^2) = p_1^2 + 4\varepsilon^2\hat{y}^2 + 4\varepsilon q$ so $\Delta_1 > \max\{p_1, 2\varepsilon\hat{y}\} \ge p_1$. Hence $\Re(r) \le 0$. Furthermore, it follows that

$$|r| = \left| \frac{2\varepsilon\hat{y}^2 + 2q + 2p_2\hat{y}i}{p_1 + \Delta} \right| \le \frac{2\varepsilon\hat{y}^2 + 2q + 2p_2|\hat{y}|}{\max\{p_1, 2\varepsilon\hat{y}\}} \le C(\hat{y} + 1). \tag{2.3}$$

From Parseval's formula,

$$\int_{-\infty}^{\infty} |D_x^m D_y^{n+1} u(x,y)|^2 \, dy = \int_{-\infty}^{\infty} r^{2m} |\hat{y}|^{2(n+1)} |\hat{h}(\hat{y})|^2 |e^{2rx}| \, d\hat{y} \le \int_{-\infty}^{\infty} r^{2m} |\hat{y}|^{2(n+1)} |\hat{h}(\hat{y})|^2 \, d\hat{y}, \tag{2.4}$$

and $D_x^m D_y^{n+1} u(x, \cdot) \in L^2(\mathbb{R})$ provided that the right-hand side of this inequality is finite.

Using (2.3) in (2.4),

$$\int_{-\infty}^{\infty} |D_x^m D_y^{n+1} u(x,y)|^2 \, dy \le C \int_{-\infty}^{\infty} (1 + |\hat{y}|)^{2m+2n+2} |\hat{h}(\hat{y})|^2 \, d\hat{y} \le C \int_{-\infty}^{\infty} |D^{m+n+1} h(y)|^2 \, dy. \tag{2.5}$$

The integrations are only over y; the solution behaves continuously with respect to x. Therefore, applying Sobolev's inequality in one dimension, for all (x,y) one has

$$|D_x^m D_y^n u(x,y)| \le C\|u\|_{H^{m+n+1}(\mathbb{R})} \le C\|h\|_{H^{m+n+1}(\mathbb{R})},$$

where the second inequality is a restatement of (2.5). This completes the proof of the Lemma. ■

Next we return to the more general situation where $h(y)$ has only the degree of smoothness at $y=0$ that is specified by (1.2). Under the additional hypothesis that $h(y)$ vanishes for $y<0$, a bound is proved for the convective derivatives of $u(x,y)$ at those points (x,y) that lie below the subcharacteristic line $y = p_1^{-1}p_2 x$ passing through the origin.

Lemma 2.2. *Let $\nu \ge -1$ be an integer. Let $h \in H^{\nu+2}(\mathbb{R}_+)$ and suppose that $h(y)=0$ for $y<0$. Suppose that either $\nu=-1$ or h satisfies (1.2). Assume that (x,y) satisfies the inequalities*

$$x > 0 \quad \text{and} \quad p_2 x - p_1 y \ge 0. \tag{2.6}$$

Then for $0<\varepsilon\le 1$ and each non-negative integer n there is a constant C_n such that the solution u of (1.1) with $f=0$ satisfies

$$|D_\beta^n u(x,y)| \le C_n(1+r^{-n+\nu+1}) \text{ for } r \le 2\varepsilon, \tag{2.7a}$$

$$|D_\beta^n u(x,y)| \le C_n\left(1+\varepsilon^{(-n+\nu+1)/2} r^{(-n+\nu+1)/2} e^{-cd^2/(\varepsilon r)} + r^{-n+\nu+1}e^{-cr/\varepsilon}\right) \text{ for } r \ge 2\varepsilon. \tag{2.7b}$$

Proof. The proof is in a series of steps.

(i) Suppose $\nu \ge 0$ and $n \le \nu$. From (1.2), $h \in H^{\nu+1}(\mathbb{R})$. Then Lemma 2.1 shows that $|D_\beta u(x,y)| \le C$ for all $n \le \nu$.

(ii) Suppose $n > \nu \ge -1$. Since $h \equiv 0$ for $y<0$, the solution formula for (1.1) is (see, e.g., [4, (3.4)])

$$u(x,y) = \frac{x}{2\pi\varepsilon}\int_0^\infty h(t)\zeta_1(t)\frac{1}{r_1}K_1(\kappa r_1(t)/(2\varepsilon))dt \quad \text{for } x>0, \tag{2.8}$$

where K_1 is a modified Bessel function of the second kind [1] and we have set $r_1(t) = \sqrt{x^2+(y-t)^2}$, $\kappa = \sqrt{p_1^2+p_2^2+4\varepsilon q}$ and $\zeta_1(t) = e^{(p_1x+p_2(y-t))/(2\varepsilon)}$. (The square integrability of h and its derivatives guarantees the convergence of this and subsequent integrals.) Define the integral operator $\mathcal{I}$ by

$$\mathcal{I}(F)(t) = \int_{t_1=t}^\infty F(t_1)dt_1.$$

Then $\nu+1$ integrations by parts in (2.8) give

$$u(x,y) = \frac{x}{2\pi\varepsilon}\int_0^\infty h^{(\nu+1)}(t)\mathcal{I}^{\nu+1}\left(\zeta_1\frac{1}{r_1}K_1(\kappa r_1/(2\varepsilon))\right)(t)\,dt,$$

where we used the property (1.2). To calculate $D_\beta^n u$, note that D_β commutes with $\mathcal{I}$ and that $D_\beta\zeta_1=0$. Setting

$$B_n = \int_0^\infty h^{(\nu+1)}(t)\mathcal{I}^{\nu+1}\left(\zeta_1 D_\beta^n\left[\frac{1}{r_1}K_1(\kappa r_1/(2\varepsilon))\right]\right)(t)\,dt,$$

we obtain

$$D_\beta^n u(x,y) = \frac{x}{2\pi\varepsilon}B_n - \frac{p_2 n}{2\pi\varepsilon}B_{n-1}. \tag{2.9}$$

It is therefore of interest to obtain bounds for B_n, and we pursue this next.

Iterations of the operator $\mathcal{I}$ give the formula

$$\mathcal{I}^{\nu+1}(F)(t_1) = \frac{1}{\nu!}\int_{t=t_1}^\infty (t-t_1)^\nu F(t)dt.$$

Using this formula in B_n,

$$B_n = \frac{1}{\nu!}\int_{t_1=0}^{\infty}\int_{t=t_1}^{\infty} h^{(\nu+1)}(t_1)(t-t_1)^{\nu}\zeta_1(t)D_\beta^n\left[\frac{1}{r_1(t)}K_1(\kappa r_1(t)/(2\varepsilon))\right]dt\,dt_1.$$

Interchange the orders of integration and take absolute values, recalling that $\nu < n$ and noting that from Sobolev's inequality, $|h^{(\nu+1)}(\cdot)| \le C\|h\|_{H^{\nu+2}(\mathbb{R}_+)}$ is bounded on $\mathbb{R}_+$. One obtains

$$|B_n| \le C\int_0^{\infty} t^{\nu+1}\zeta_1(t)\left|D_\beta^n\left[\frac{1}{r_1(t)}K_1(\kappa r_1(t)/(2\varepsilon))\right]\right| dt. \tag{2.10}$$

To calculate the integrand we will use the formula [1, (9.6.28)]

$$\left(\lambda^{-1}\frac{d}{d\lambda}\right)^n\left[\lambda^{-1}K_1(\lambda)\right] = (-1)^n\lambda^{-n-1}K_{n+1}(\lambda). \tag{2.11}$$

Set $\lambda = \kappa r_1/(2\varepsilon)$ and define the operator $\mathcal{D}(\cdot) = \lambda^{-1}D_\lambda(\cdot)$. Noting that $r_1^{-1} = (\kappa/(2\varepsilon))\lambda^{-1}$, $\lambda_x = \kappa x/(2\varepsilon r_1)$, $\lambda_y = \kappa(y-t)/(2\varepsilon r_1)$, and setting $\mu(x,y) = -p_2 x + p_1(y-t)$, we get

$$\begin{aligned} D_\beta\mu &= |\alpha|^2,\\ D_\beta &= \frac{\kappa}{2\varepsilon}\frac{\mu}{r_1}D_\lambda = \frac{\kappa^2}{4\varepsilon^2}\mu\mathcal{D},\\ D_\beta\left[\frac{1}{r_1}K_1(\kappa r_1/(2\varepsilon))\right] &= \frac{\kappa^3}{8\varepsilon^3}\mu\mathcal{D}\big(\lambda^{-1}K_1(\lambda)\big). \end{aligned}$$

Let $[j,k,\ell]$ denote a term of the form $C\varepsilon^{-j}\mu^k\mathcal{D}^\ell\big(\lambda^{-1}K_1(\lambda)\big)$, where j,k and ℓ are non-negative integers. Then

$$D_\beta[j,k,\ell] = k[j,k-1,\ell] + [j+2,k+1,\ell+1]. \tag{2.12}$$

If $k=0$, the first term on the right hand side of (2.12) is not present. The following general formula is easily proved by induction:

$$D_\beta^n\left[\frac{1}{r_1}K_1(\kappa r_1/(2\varepsilon))\right] = \sum_{\ell=0}^{\lfloor n/2\rfloor}[2n-2\ell+1,\ n-2\ell,\ n-\ell],$$

where $\lfloor n/2\rfloor$ denotes the largest integer m that satisfies $m \le n/2$. Applying this formula in (2.10) then invoking (2.11), we obtain

$$|B_n| \le C\sum_{\ell=0}^{\lfloor n/2\rfloor}\varepsilon^{-(2n-2\ell+1)}\int_0^{\infty} t^{\nu+1}|\mu|^{n-2\ell}(r_1/(2\varepsilon))^{-(n-\ell+1)}\zeta_1(t)K_{n-\ell+1}(\kappa r_1/(2\varepsilon))\,dt.$$

We make the change of variable $t = 2\varepsilon\tau$ in the integrals. Set $\xi = x/(2\varepsilon)$, $\eta = y/(2\varepsilon)$, $\rho_1 = \sqrt{\xi^2+(\eta-\tau)^2}$ and $\rho = \sqrt{\xi^2+\eta^2}$, so $r = 2\varepsilon\rho$. Let $\tilde\mu = -p_2\xi + p_1(\eta-\tau)$, so $\mu = 2\varepsilon\tilde\mu$. We obtain

$$|B_n| \le C\varepsilon^{-n+\nu+1}\sum_{\ell=0}^{\lfloor n/2\rfloor}\int_0^{\infty}\tau^{\nu+1}|\tilde\mu|^{n-2\ell}\rho_1^{-(n-\ell+1)}e^{p_1\xi+p_2(\eta-\tau)}K_{n-\ell+1}(\kappa\rho_1)\,d\tau. \tag{2.13}$$

The remainder of the proof divides into two cases: $r \le 2\varepsilon$ and $r \ge 2\varepsilon$.

(iii) Suppose $r \ge 2\varepsilon$. Here, $\rho = \sqrt{\xi^2+\eta^2} \ge 1$. If $\xi \ge \min\{1/\sqrt{2}, p_1/|\alpha|\}$, then clearly $\rho_1 \ge \min\{1/\sqrt{2}, p_1/|\alpha|\}$. If $\xi < \min\{1/\sqrt{2}, p_1/|\alpha|\}$, we claim that $\eta < 0$: for if not, then the inequality $\eta \le p_2\xi/p_1$ of (2.6) implies that $\eta^2 \le p_2^2\xi^2/p_1^2$ so $\xi^2+\eta^2 \le (p_1^2+p_2^2)\xi^2/p_1^2 = |\alpha|^2\xi^2/p_1^2 < 1$, a contradiction; furthermore $\eta^2 \ge 1-\xi^2 > 1/2$ so $|\tau-\eta| = \tau+|\eta| \ge 1/\sqrt{2}$ and $\rho_1 > 1/\sqrt{2}$. Hence in either event $\kappa\rho_1 \ge c > 0$, so we may use the inequality

$K_j(\kappa\rho_1) \le C\rho_1^{-1/2} e^{-\kappa\rho_1}$ in (2.13). (See [1, (9.7.2)] for the corresponding asymptotic formula.) With this, we obtain

$$|B_n| \le C\varepsilon^{-n+\nu+1} \sum_{\ell=0}^{\lfloor n/2 \rfloor} \int_0^\infty \tau^{\nu+1} |\tilde{\mu}|^{n-2\ell} \rho_1^{-(n-\ell+3/2)} \zeta(\tau)\, d\tau, \tag{2.14}$$

where $\zeta(\tau) = e^{-(\kappa\rho_1 - p_1\xi - p_2(\eta-\tau))}$

To estimate the integrals in (2.14) two inequalities are needed. Setting $\rho_2 = \sqrt{\xi^2+s^2}$ one has

$$\begin{aligned} \kappa\rho_2 - p_1\xi + p_2 s &= \frac{\kappa^2\rho_2^2 - (p_1\xi - p_2 s)^2}{\kappa\rho_2 + p_1\xi - p_2 s} \\ &= \frac{(p_1^2 + p_2^2 + 4\varepsilon q)(\xi^2+s^2) - p_1^2\xi^2 - p_2 s^2 + 2p_1p_2\xi s}{\kappa\rho_2 + p_1\xi - p_2 s} \\ &= \frac{(p_2\xi + p_1 s)^2}{\kappa\rho_2 + p_1\xi - p_2 s} + \frac{4\varepsilon q(\xi^2+s^2)}{\kappa\rho_2 + p_1\xi - p_2 s}. \end{aligned}$$

The Cauchy-Schwarz inequality yields $p_1\xi - p_2 s < \kappa\rho_2$, so

$$0 < \kappa\rho_2 + p_1\xi - p_2 s \le 2\kappa\rho_2.$$

Therefore

$$\kappa\rho_2 - p_1\xi + p_2 s \ge \frac{(p_2\xi + p_1 s)^2}{2\kappa\rho_2}.$$

Replacing s by $\tau - \eta$ and noting that ρ_2 then becomes ρ_1, we obtain

$$\kappa\rho_1 - p_1\xi + p_2(\tau-\eta) \ge \frac{(p_2\xi + p_1(\tau-\eta))^2}{2\kappa\rho_1}. \tag{2.15}$$

Also, since $p_2\xi \ge p_1\eta$, $(p_2\xi + p_1(\tau-\eta))^2 = ((p_2\xi - p_1\eta) + p_1\tau)^2 \ge (p_2\xi - p_1\eta)^2 + p_1^2\tau^2$. Using this inequality in (2.15) and setting $\delta = |p_2\xi - p_1\eta|$, we get

$$\kappa\rho_1 - p_1\xi + p_2(\tau-\eta) \ge \frac{\delta^2}{2\kappa\rho_1} + \frac{p_1^2\tau^2}{2\kappa\rho_1}. \tag{2.16}$$

Both (2.15) and (2.16) will be used in what follows. Note that $\delta/|\alpha|^2$ is the distance from (ξ,η) to the line $p_1\eta = p_2\xi$ in the $\xi\eta$-plane.

In (2.14) write $\zeta = \zeta^{1/2}\zeta^{1/2}$. Use (2.15) to bound one of the factors $\zeta^{1/2}$ and (2.16) to bound the other factor. This yields

$$\zeta(\tau) \le e^{-c(p_2\xi + p_1(\tau-\eta))^2/\rho_1} e^{-c(\delta^2+\tau^2)/\rho_1}.$$

Now use the inequality

$$|\tilde{\mu}^a| e^{-c(p_2\xi + p_1(\tau-\eta))^2/\rho_1} \le C\rho_1^{a/2}$$

to obtain

$$|B_n| \le C\varepsilon^{-n+\nu+1} \int_0^\infty \tau^{\nu+1} \rho_1^{-(n+3)/2} e^{-c(\delta^2+\tau^2)/\rho_1}\, d\tau.$$

Hence $|B_n| \le B_{n,1} + B_{n,2}$ where the two terms refer to integrations over $(0,\rho)$ and (ρ,∞) respectively.

If $\eta > 0$, since $p_2\xi \ge p_1\eta$ one has ξ bounded below by $C\rho$. Hence for all $\tau \ge 0$ and all η,

$$\begin{aligned} \sqrt{2}\rho_1 &\ge \xi + |\tau - \eta| \\ &\ge \begin{cases} C\rho + |\tau-\eta| & \text{if } \eta > 0 \\ \xi + \tau + |\eta| & \text{if } \eta \le 0 \end{cases} \\ &\ge C\rho. \end{aligned} \tag{2.17}$$

First consider $B_{n,1}$. For $\tau \le \rho$, one has $\rho_1 \le \xi + |\tau - \eta| \le \xi + \rho + |\eta| \le C\rho$. Invoking this inequality and (2.17),

$$B_{n,1} = C\varepsilon^{-n+\nu+1} \int_0^\rho \tau^{\nu+1} \rho^{-(n+3)/2} e^{-c(\delta^2+\tau^2)/\rho}\, d\tau = C\varepsilon^{-n+\nu+1} \rho^{-(n+3)/2} e^{-c\delta^2/\rho} \int_0^\rho \tau^{\nu+1} e^{-c\tau^2/\rho}\, d\tau.$$

Making the change of variable $z = \tau^2/\rho$ we obtain

$$\begin{aligned} B_{n,1} &= C\varepsilon^{-n+\nu+1} \rho^{(\nu-n-1)/2} e^{-c\delta^2/\rho} \int_0^\rho z^{\nu/2} e^{-cz}\, dz \\ &\le C\varepsilon^{-n+\nu+1} \rho^{(\nu-n-1)/2} e^{-c\delta^2/\rho} \\ &= C\varepsilon^{(-n+\nu+3)/2} r^{(-n+\nu-1)/2} e^{-c\delta^2/(\varepsilon r)}. \end{aligned} \tag{2.18}$$

Next consider $B_{n,2}$. For $\tau \ge \rho$ one has $\rho_1 \le \xi + \tau + |\eta| \le \sqrt{2}\rho + \tau \le C\tau$, so $e^{-c\tau^2/\rho_1} \le e^{-c\tau}$. Invoking this inequality and (2.17),

$$\begin{aligned} B_{n,2} &\le C\varepsilon^{-n+\nu+1} \int_\rho^\infty \tau^{\nu+1} \rho^{-(n+3)/2} e^{-c\tau}\, d\tau \\ &\le C\varepsilon^{-n+\nu+1} \rho^{-(n+3)/2} \rho^{\nu+1} e^{-c\rho} \\ &= C\varepsilon^{(3-n)/2} r^{\nu-(n+1)/2} e^{-cr/\varepsilon} \\ &\le C\varepsilon r^{-n+\nu} e^{-cr/\varepsilon}, \end{aligned} \tag{2.19}$$

since $(r/\varepsilon)^{n/2} e^{-cr/\varepsilon} \le C$ and $\varepsilon^{1/2} \le (r/2)^{1/2}$. From (2.18) and (2.19),

$$|B_n| \le B_{n,1} + B_{n,2} \le C\varepsilon^{(-n+\nu+3)/2} r^{(-n+\nu-1)/2} e^{-cd^2/(\varepsilon r)} + C\varepsilon r^{-n+\nu} e^{-cr/\varepsilon}.$$

Therefore, recalling (2.9),

$$\begin{aligned} |D_\beta^n u(x,y)| &\le C(x\varepsilon^{-1} B_n + \varepsilon^{-1} B_{n-1}) \\ &\le C\varepsilon^{(-n+\nu+1)/2} r^{(-n+\nu+1)/2} e^{-cd^2/(\varepsilon r)} + Cr^{-n+\nu+1} e^{-cr/\varepsilon} \\ &\quad + C\varepsilon^{(-n+\nu+2)/2} r^{(-n+\nu)/2} e^{-cd^2/(\varepsilon r)} + Cr^{-n+\nu+1} e^{-cr/\varepsilon} \\ &\le C\varepsilon^{(-n+\nu+1)/2} r^{(-n+\nu+1)/2} e^{-cd^2/(\varepsilon r)} + Cr^{-n+\nu+1} e^{-cr/\varepsilon}, \end{aligned}$$

since $\varepsilon^{1/2} \le (r/2)^{1/2}$. This proves (2.7b).

(iv) Suppose $r \le 2\varepsilon$. In this case, $\rho \le 1$. Using (2.13) we write $B_n \le B_{n,1} + B_{n,2}$ where the two terms refer to integrations over $(0,4)$ and $(4,\infty)$ respectively. Now $|\tilde{\mu}|^{n-2\ell} \le C\rho_1^{n-2\ell}$ since $n - 2\ell \ge 0$ in the summation of (2.13), so it follows that

$$B_{n,1} \le C\varepsilon^{-n+\nu+1} \sum_{\ell=0}^{\lfloor n/2 \rfloor} \int_0^4 \tau^{\nu+1} \rho_1^{-\ell-1} e^{p_1\xi + p_2(\eta-\tau)} K_{n-\ell+1}(\kappa\rho_1)\, d\tau$$

and

$$B_{n,2} \le C\varepsilon^{-n+\nu+1} \sum_{\ell=0}^{\lfloor n/2 \rfloor} \int_4^\infty \tau^{\nu+1} \rho_1^{-\ell-1} e^{p_1\xi + p_2(\eta-\tau)} K_{n-\ell+1}(\kappa\rho_1)\, d\tau. \tag{2.20}$$

Consider first the bound on $B_{n,1}$. When $\tau \in (0,4)$, one has $\rho_1 \le C$ so $K_{n-\ell+1}(\kappa\rho_1) \le C\rho_1^{-n+\ell-1}$ (see [1, (9.6.9)] for the corresponding asymptotic formula). As $e^{p_1\xi+p_2(\eta-\tau)} \le C$, we have

$$B_{n,1} \le C\varepsilon^{-n+\nu+1} \int_0^4 \tau^{\nu+1} \rho_1^{-n-2} d\tau.$$

Now $\rho_1 \geq c(\xi + |\tau - \eta|)$. If $\eta < 0$ one has $\rho_1 \geq c(\xi + |\eta| + \tau) \geq c(\rho + \tau)$. If $\eta > 0$, then using (2.6) one has $\eta < C\xi$ so $\xi \geq c\rho$; hence $\rho_1 \geq c(\rho + |\eta - \tau|)$. Therefore in either event we get $\rho_1 \geq c(\rho + |b - \tau|)$ where $0 \leq b \leq \eta \leq \rho$. Since $\nu < n$,

$$
\begin{aligned}
B_{n,1} &\leq C\varepsilon^{-n+\nu+1} \int_0^\infty \frac{\tau^{\nu+1}}{(\rho + |b-\tau|)^{n+2}} d\tau \\
&\leq C\varepsilon^{-n+\nu+1} \rho^{-n+\nu} \\
&\leq C\varepsilon r^{-n+\nu}. \qquad (2.21)
\end{aligned}
$$

Next consider $B_{n,2}$. For $\tau \geq 4$ one has $\rho_1 \geq \tau - |\eta| \geq 1$, so in (2.20) we can use the inequality $K_{n-\ell+1}(\kappa\rho_1) \leq C\rho_1^{-1/2} e^{-\kappa\rho_1}$. Also invoking (2.16) we obtain

$$
\begin{aligned}
B_{n,2} &\leq C\varepsilon^{-n+\nu+1} \sum_{\ell=0}^{\lfloor n/2 \rfloor} \int_4^\infty \tau^{\nu+1} \rho_1^{-\ell-3/2} e^{-(\kappa\rho_1 - p_1\xi - p_2(\eta-\tau))} d\tau \\
&\leq C\varepsilon^{-n+\nu+1} \int_4^\infty \tau^{\nu+1} \rho_1^{-3/2} e^{-c\tau^2/\rho_1} d\tau \\
&= C\varepsilon^{-n+\nu+1} \int_4^\infty \tau^{\nu-2} \left(\frac{\tau^2}{\rho_1}\right)^{3/2} e^{-c\tau^2/\rho_1} d\tau \\
&\leq C\varepsilon^{-n+\nu+1} \int_4^\infty \tau^{\nu-2} e^{-c\tau^2/\rho_1} d\tau \\
&\leq C\varepsilon^{-n+\nu+1} \int_4^\infty \tau^{\nu-2} e^{-c\tau} d\tau \\
&\leq C\varepsilon^{-n+\nu+1}, \qquad (2.22)
\end{aligned}
$$

where the penultimate inequality follows from $\rho_1 \leq \xi + \tau + |\eta| \leq 2 + \tau \leq 2\tau$.

From (2.21) and (2.22),

$$|B_n| \leq C\varepsilon r^{-n+\nu} + C\varepsilon^{-n+\nu+1} \leq C\varepsilon r^{-n+\nu}$$

as $r \leq 2\varepsilon$ and $n > \nu$. Consequently

$$|D_\beta^n u| \leq C(x\varepsilon^{-1} B_n + \varepsilon^{-1} B_{n-1}) \leq C r^{-n+\nu+1} \text{ for } r \leq 2\varepsilon.$$

This completes the proof of (2.7a) and of the lemma. ∎

It will be noted that in Lemma 2.2 the order of the derivative being estimated does not depend on the regularity assumed by h. This is possible because the vanishing of h for $y < 0$ and the condition (2.6) imply that the derivatives are being estimated in a region associated with a zero boundary condition.

Now we give the main result of this section. In it, the inequalities (2.7) are established without requiring that h satisfy (2.6).

Theorem 2.1. *Let n be a non-negative integer. Let $h \in H^{n+1}(\mathbb{R}_\pm)$. Let $\nu \geq -1$ be an integer with $\nu < n$ and suppose that either $\nu = -1$ or h satisfies (1.2). Then there is a constant C_n such that for $0 < \varepsilon \leq 1$ the solution u of (1.1) with $f = 0$ satisfies the inequalities (2.7a) and (2.7b).*

Proof. To start, note that if $\nu = -1$ it suffices to prove the theorem in the two cases $h(y) = 0$ on $(-\infty, 0)$ and $h(y) = 0$ on $(0, \infty)$, since the general result then follows from the linearity of the problem. The proofs of these two cases are similar so one can assume that

$$h(y) = 0 \text{ for } y < 0. \qquad (2.23)$$

If $\nu \geq 0$ it also suffices to prove the theorem under the assumption (2.23), as the following argument shows. Let $h_a \in H^{n+1}(\mathbb{R})$ be a function with $D^k h_a(0) = D^k h(\pm 0)$ for $k = 0, \cdots, \nu$, and let u_a be the solution to the problem $Lu_a = 0$ for $x > 0$, $u_a(0, y) = h_a(y)$. From Lemma 2.1, $|D^k u_a(x, y)| \leq C$ for $k = 0, \cdots, n$ and all (x, y) such that $x > 0$. Therefore it suffices to prove that the function $u - u_a$ satisfies the inequalities (2.7a) and (2.7b). Thus we may assume that h satisfies $D^k h(\pm 0) = 0$ for $k = 0, \cdots, \nu$. We then write $h = h_+ + h_-$ where $h_+ = 0$ when $y \leq 0$ and $h_- = 0$ when $y \geq 0$, and again the proof has been reduced to the proof under the assumption (2.23).

Lemma 2.2 yields (2.7) immediately for all (x, y) such that $x > 0$ and $p_2 x - p_1 y \geq 0$. It remains to prove (2.7) for all (x, y) such that $x > 0$ and $p_2 x - p_1 y < 0$. Let $h_1 \in H^{n+1}(\mathbb{R})$ be an extension of h from $[0, \infty)$ to $\mathbb{R}$. Let u_1 be the solution of the problem

$$Lu_1 = 0 \text{ for } x > 0, \quad u_1(0, y) = h_1(y) \text{ for } y \in (\infty, \infty).$$

By Lemma 2.1 we have

$$|D_\beta^n u_1(x, y)| \leq C \text{ for } x > 0. \tag{2.24}$$

Let $u_2 = u - u_1$, $h_2 = h - h_1$, so $h_2(y) = 0$ for $y > 0$. Let u_3 and h_3 be defined by

$$u_3(x, y) = u_2(x, -y), \quad h_3(y) = h_2(-y).$$

Then u_3 is the solution of the problem

$$-\varepsilon \Delta u_3 + p_1 u_{3,x} - p_2 u_{3,y} + q u_3 = 0 \text{ for } x > 0, \qquad u_3(0, y) = h_3(y) \text{ for } y \in (-\infty, \infty).$$

By its definition, $h_3(y) = 0$ for $y < 0$. Applying Lemma 2.2 to u_3 while noting that the coefficient of $u_{3,y}$ in the above differential equation is $-p_2$, we see that $|(p_2 D_x + p_1 D_y)^n u_3(x, y)|$ is bounded by the right-hand sides of (2.7) for $x > 0$ and $-p_2 x - p_1 y \geq 0$. Hence $D_\beta^n u_2$ satisfies (2.7) for all (x, y) such that $x > 0$ and $-p_2 x + p_1 y \geq 0$. Using the triangle inequality and (2.24), it follows that $D_\beta^n u$ satisfies (2.7) for all (x, y) with $-p_2 x + p_1 y \geq 0$. This completes the proof of the theorem. ■

3 Bounds on all derivatives

This section contains the proof of Theorem 1.1. One can subtract from u the solution $\tilde{u}$ of the problem $L\tilde{u} = f$ for $x > 0$, $\tilde{u}(0, y) = 0$ for $y \in \mathbb{R}$, and apply Lemma 2.1 to bound the derivatives of $\tilde{u}$. Thus it suffices to prove Theorem 1.1 in the case $f = 0$.

Note that the inequality (1.3b) of Theorem 1.1 is slightly different from the inequality (2.7b) of Lemma 2.2 and Theorem 2.1 in two respects: in (1.3b) there appears the factor $e^{-cd^2/\varepsilon}$, whereas the corresponding factor in (2.7b) is $e^{-cd^2/(\varepsilon r)}$; also, the inequality (1.3b) is asserted only for a bounded range of r, which is (arbitrarily) taken to be $r \leq 1$. Since $e^{-cd^2/(\varepsilon r)} \leq e^{-cd^2/\varepsilon}$ for $r \leq 1$, the inequality (2.7b) implies (1.3b) if $m = 0$. The proof of Theorem 1.1 is by induction on m; the case $m = 0$ is covered in Theorem 2.1. Let M be a non-negative integer, and assume (1.3) holds true for $m = M$ and all $n \geq 0$.

One has $D_\alpha^2 + D_\beta^2 = |\alpha|^2(D_x^2 + D_y^2)$, so the differential equation is $-\varepsilon|\alpha|^{-2}(D_\alpha^2 u + D_\beta^2 u) + D_\alpha u + qu = 0$. Let (x, y) be given with $x > 0$. For $s > 0$, define $w(s) = D_\alpha^M D_\beta^n u(x + p_1 s, y + p_2 s)$ and $F(s) = (\varepsilon D_\alpha^M D_\beta^{n+2} u + |\alpha|^2 q D_\alpha^M D_\beta^n u)(x + p_1 s, y + p_2 s)$. The differential equation gives

$$-\varepsilon w'' + |\alpha|^2 w' = F. \tag{3.1}$$

The proof is now divided into two cases.

The case $r \geq 2\varepsilon$. Using the integrating factor $e^{-|\alpha|^2 s/\varepsilon}$, one obtains

$$w'(s) = w'(1) e^{-|\alpha|^2(1-s)/\varepsilon} + \varepsilon^{-1} \int_s^1 F(t) e^{-|\alpha|^2(t-s)/\varepsilon} dt. \tag{3.2}$$

Integrating (3.2) over $(0, 1)$ and then solving for $w'(1)$, one obtains

$$w'(1) = \frac{|\alpha|^2}{\varepsilon(1 - e^{-|\alpha|^2/\varepsilon})} \left[w(1) - w(0) - \varepsilon^{-1} \int_{s=0}^1 \int_{t=s}^1 F(t) e^{-|\alpha|^2(t-s)/\varepsilon} dt ds \right]. \tag{3.3}$$

Taking $s = 0$ in (3.2) gives

$$w'(0) = w'(1)e^{-|\alpha|^2/\varepsilon} + \varepsilon^{-1} \int_0^1 F(t)e^{-|\alpha|^2 t/\varepsilon} dt.$$

Using (3.3) to replace $w'(1)$ in this formula, one obtains

$$\begin{aligned} w'(0) \quad &= \quad \frac{|\alpha|^2 e^{-|\alpha|^2/\varepsilon}}{\varepsilon(1 - e^{-|\alpha|^2/\varepsilon})} \left[w(1) - w(0) - \varepsilon^{-1} \int_{s=0}^1 \int_{t=s}^1 F(t)e^{-|\alpha|^2(t-s)/\varepsilon} dtds \right] \\ &\quad + \varepsilon^{-1} \int_0^1 F(t)e^{-|\alpha|^2 t/\varepsilon} dt. \end{aligned}$$

Taking $\max |F|$ out of the integrals yields

$$|D_\alpha^{M+1} D_\beta^n u(x,y)| = |w'(0)| \le C\big[|w(0)| + |w(1)| + \max_{s\in[0,1]} |F(s)|\big]. \tag{3.4}$$

Let $r(s) = \sqrt{(x + p_1 s)^2 + (y + p_2 s)^2}$, so $r(0) = r$. For $s \ge 0$, one can show that $r(s) \ge cr$. Since $\nu + 1 \le n$, the bounds on the right hand sides of (1.3b) are bounded by a constant times the corresponding bounds at $s = 0$. (This observation uses the fact that the distance $d(s)$ from the point $(x + p_1 s, y + p_2 s)$ to the line $p_1 y = p_2 x$ is independent of s.) Therefore, using the inductive assumption, if $r \ge 2\varepsilon$

$$\begin{aligned} |D_\alpha^{M+1} D_\beta^n u(x,y)| &\le C(|w(0| + |w(1)| + |F(0)|) \\ &\le C\left(1 + \varepsilon^{(-n+\nu+1)/2} r^{-M+(-n+\nu+1)/2} e^{-cd^2/\varepsilon} + r^{-M-n+\nu+1} e^{-cr/\varepsilon}\right) \\ &\quad + C\varepsilon\left(1 + \varepsilon^{(-n-2+\nu+1)/2} r^{-M+(-n-2+\nu+1)/2} e^{-cd^2/\varepsilon} + r^{-M-n-2+\nu+1} e^{-cr/\varepsilon}\right) \\ &= C\left(1 + \varepsilon^{(-n+\nu+1)/2} r^{-M+(-n+\nu+1)/2} e^{-cd^2/\varepsilon} + r^{-M-n+\nu+1} e^{-cr/\varepsilon}\right) \\ &\quad + C\left(\varepsilon + \varepsilon^{(-n+\nu+1)/2} r^{-(M+1)+(-n+\nu+1)/2} e^{-cd^2/\varepsilon} + \varepsilon r^{-1} r^{-(M+1)-n+\nu+1} e^{-cr/\varepsilon}\right). \end{aligned}$$

Since $2\varepsilon \le r \le 1$ this inequality gives (1.3b) with $m = M + 1$, and so completes the inductive proof of (1.3b).

The case $r < 2\varepsilon$. Let $s^* > 0$ be the smallest number such that $r(s^*) = 2\varepsilon$. Using the integrating factor $e^{-|\alpha|^2 s/\varepsilon}$ and integrating from 0 to s^* one obtains

$$w'(0) = e^{-|\alpha|^2 s^*/\varepsilon} w'(s^*) + \varepsilon^{-1} \int_0^{s^*} e^{-|\alpha|^2 t/\varepsilon} F(t) dt.$$

Consequently

$$|w'(0)| \le |w'(s^*)| + \varepsilon^{-1} \int_0^{s^*} |F(t)| dt.$$

For $|w'(s^*)|$ we use the bound in (1.3b) and for $|F(t)|$ we use the inductive assumption. Since $r(t) \ge c(r + t)$ we get

$$\begin{aligned} |w'(0)| &\le Cr^{-M-n+\nu+1} + C\varepsilon^{-1} \int_0^{s^*} [\varepsilon(r+t)^{-M-(n+2)+\nu+1} + (r+t)^{-M-n+\nu+1}] dt \\ &\le Cr^{-M-n+\nu+1} + C[r^{-M-(n+2)+\nu+2} + \varepsilon^{-1} r^{-M-n+\nu+2}] \\ &= Cr^{-M-n+\nu+1} + Cr^{-(M+1)-n+\nu+1} + C\varepsilon^{-1} r^2 \cdot r^{-(M+1)-n+\nu+1}. \end{aligned}$$

Since $r \le 2\varepsilon \le 2$, $\varepsilon^{-1} r^2 \le 2r \le 4$. Therefore $|D_\alpha^{M+1} D_\beta^n u(x,y)| = |w'(0)| \le Cr^{-(M+1)-n+\nu+1}$, which gives (1.3a) with $m = M + 1$, and so completes the inductive proof of (1.3a). ∎

References

[1] M. Abramowitz and I. Stegun, Handbook of mathematical functions, National Bureau of Standards, 1965.

[2] W. Dörfler, Uniform a priori estimates for singularly perturbed elliptic equations in multidimensions, SIAM J. Numer. Anal. 36 (1999), 1878–1900.

[3] W. Hundsdorfer and J.G. Verwer, Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations, Springer-Verlag, Berlin, 2003.

[4] R.B. Kellogg and M. Stynes, Corner singularities and boundary layers in a simple convection-diffusion problem. Report of the IMI, 03:13, University of South Carolina, www.math.sc.edu/ imip03.html.

[5] T. Linß and M. Stynes, Asymptotic analysis and Shishkin-type decomposition for an elliptic convection-diffusion problem, J. Math. Anal. Applic. 261 (2001), 604-632.

[6] H.-G. Roos, Optimal uniform convergence of basic schemes for elliptic problems with strong parabolic boundary layers, J. Math. Anal. Applic. 267 (2002), 194–208.

[7] H.-G. Roos, M. Stynes and L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Volume 24, Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1996.

[8] S.-D. Shih and R. B. Kellogg, Asymptotic analysis of a singular perturbation problem, SIAM J. Math. Anal. 18 (1987), 1467–1511.

[9] G. I. Shishkin, Grid approximation of singularly perturbed elliptic and parabolic equations. Second Doctoral Thesis, Keldysh Institute, Moscow, 1990.