Decoding in Compressed Sensing

Ronald DeVore

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- Here good means the samples $y = \Phi x$ contain enough information to approximate x well

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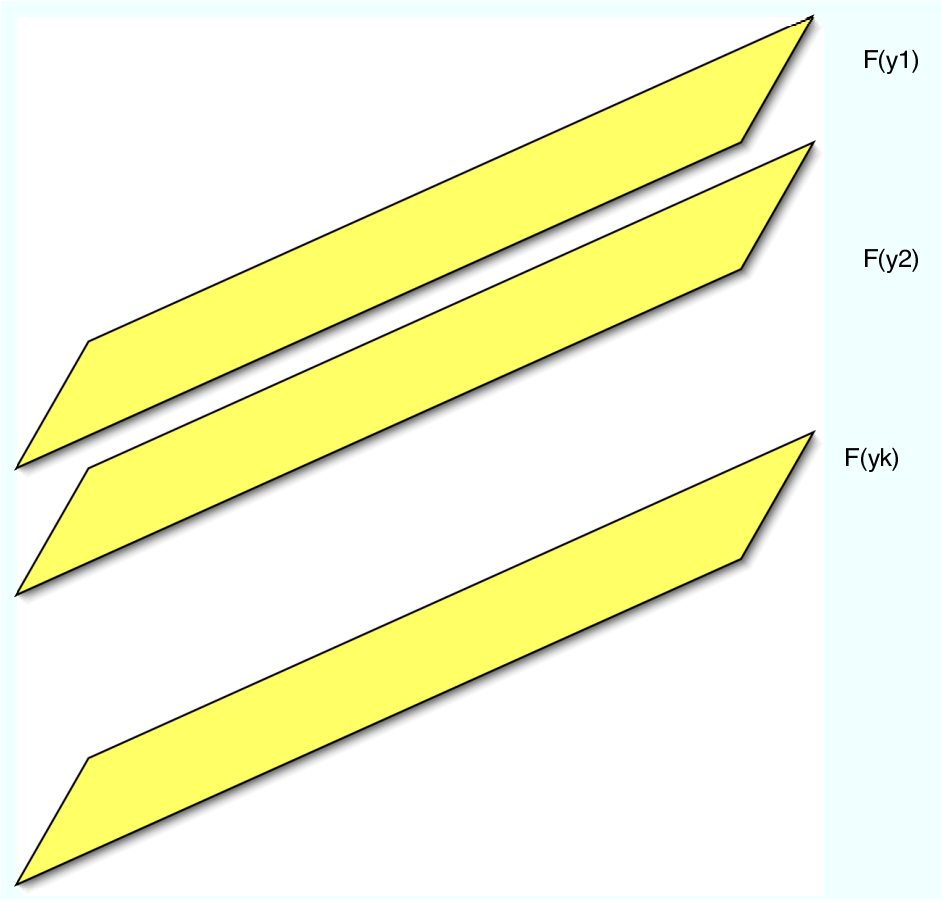

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- Let $\mathcal{A} := \mathcal{A}_{n,N} := \{(\Phi, \Delta) : \ \Phi \ \text{ is } \ n \times N\}$

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- Given an encoding - decoding pair (Φ, Δ), we say that this pair is Instance-Optimal of order k for ℓ_p if for an absolute constant $C > 0$

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- The encoding - decoding pairs which have the largest k are the best.

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- Theorem of Cohen-Dahmen-DeVore: We have instance optimality of order k in ℓ_p for some decoder Δ if and only if Φ has the Null Space Property (NSP) of order $2k$: for each $\#(T) \leq 2k$ we have

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- Here the decoder is pushed into the background: The above theorem gives no information about whether practical decoders will work

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- CDD show RIP is sufficient to guarantee NSP but the exact results depend very strongly on p

Sample Results: ℓ_1

- If Φ has RIP for $2k$ then Φ is instance-optimal of order k in ℓ_1:

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- This range of k cannot be improved: from now on we refer to this as the largest range of k

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- This bound cannot be improved
- Matrices that satisfy instance-optimal for this range of k are obtained from matrices which satisfy RIP for $\bar{k} = k^{2/p-1} N^{2-2/p}$

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- Problem: None of these are constructive. Can we put our hands on matrices??
- No constructions are known for largest range of k

Instance-Optimality in Probability

- We saw that Instance-Optimality for ℓ_2^N is not viable
- Suppose $\Phi(\omega)$ is a collection of random matrices
- We say this family satisfies RIP of order k with probability $1-\epsilon$ if a random draw $\{\Phi(\omega)\}$ will satisfy RIP of order k with probability $1-\epsilon$
- We say $\{\Phi(\omega)\}$ is bounded with probability $1-\epsilon$ if given any $x \in I\!R^N$ with probability $1-\epsilon$ a random draw $\{\Phi(\omega)\}$ will satisfy

$$\|\Phi(\omega)(x)\|_{\ell_2^N} \leq C_0 \|x\|_{\ell_2^N}$$

  with C_0 an absolute constant
- Our earlier analysis showed that Gaussian and Bernouli random matrices have these properties with $\epsilon = e^{-cn}$

Theorem: Cohen-Dahmen-DeVore

- If $\{\Phi(\omega)\}$ satisfies RIP of order $3k$ and boundedness each with probability $1-\epsilon$ then there are decoders $\Delta(\omega)$ such that given any $x \in \ell_2^N$ we have with probability $1-2\epsilon$

$$\|x - \Delta(\omega)\Phi(\omega)(x)\|_{\ell_2^N} \leq C_0 \sigma_k(x)_{\ell_2^N}$$

- Instance-optimality in probability
- Range of k is $k \leq c_0 n/\log(N/n)$
- Decoder is impractical

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- We shall make some remarks on these decoders emphasizing the last one

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- Number of computations to decode?
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- Theorems versus numerical examples
- Instance Optimality in Probability
- Given that we cant construct best encoding matrices it seems that the best results would correspond to random draws of matrices

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- Recall that x is an ℓ_1 minimizer if and only if

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- then all $x \in \Sigma_k$ have unique ℓ_1 minimizers equal to x

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- x^i is $z^i_{j_1}, \cdots, z^i_{j_i}$ augmented by zeros in the other coordinates, $x^i_j = z^i_j, \quad \text{if} \quad j \in \{j_1, \cdots, j_i\}, \quad 0 \quad \text{otherwise.}$

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- We can again solve for $x(w)$ in $O(Nn^2)$ operations

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- Unique minimizer for these problems shows $x(w) = x^*$

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- If $z \in I\!R^N$ we let $r(z)_K$ be its K-th largest entry (in absolute value)

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- The main result we prove is the following theorem

Theorem

Let $k \geq 1$ and define $K = k + 6$. We assume that Φ satisfies the Null Space Property for ℓ_1 of order $3K$ for $\gamma \leq 1/2$. Then, for each $x \in I\!R^N$, $y = \Phi(x)$, the Algorithm converges and its limit $\bar{x}$ satisfies

$$\|x - \bar{x}\|_{\ell_1} \leq C_1 \sigma_k(x)_{\ell_1}, \quad C_1 := \frac{5(1+\gamma)}{1-\gamma}.$$

In particular if x is k-sparse then x^m converges to x.

Exponential Convergence

- A second theorem shows that the algorithm converges exponentially to x^* if $x^* \in \Sigma_k$ and the starting point is close enough

  Theorem For a given $0 < \rho < 1$, assume Φ satisfies NSP of order $3K$ with constant γ such that

  $\mu := \frac{\gamma}{1-\rho}\left(1 + \frac{1}{K-k}\right) < 1$. Let m_0 be such that

  $$\|x^{m_0} - x^*\|_{\ell_1} \leq R^* := \rho \min_{i \in T} |x_i| = \rho\, r(x)_k.$$

  Then for all $m \geq m_0$, we have

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- This is observed in practice but only proved for one or two supported vectors

Theorem Assume Φ satisfies NSP of order $3K$ with constant γ sufficiently small. Then for any vector x^* which has support $k = 1, 2$ there is an absolute constant C_0 and a $\rho < 1$ such that

$$\|x^m - x^*\|_{\ell_1} \leq C_0 \rho^m \|x\|_{\ell_1}, \quad m = 1, 2, \ldots$$

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- Before embarking on the proof we want to bring out a certain geometric result on ℓ_1 minimization which we shall utilize
- As a preliminary we consider the operation of rearrangements
- We define $r(z)$ as the rearrangement of the sequence $|z_j|$ into decreasing order. In other words $r(z)_j$ is the j-th largest of the $|z_\nu|$

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- For any $J > j$, we have

$$(J - j)r(z)_J \leq \|z - z'\|_{\ell_1} + \sigma_j(z')_{\ell_1}$$

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- Reverse the roles of z and z'
- Next we approximate z by the j-term approximation u of z' in ℓ_1

$$\sigma_j(z)_{\ell_1} \leq \|z - u\|_{\ell_1} \leq \|z - z'\|_{\ell_1} + \sigma_j(z')_{\ell_1},$$

- Again reverse roles of z, z'
- $(J - j) r(z)_J \leq \sigma_j(z)_{\ell_1}$

A Geometric Property

- Assume that NSP holds for some k and $\gamma < 1$

A Geometric Property

- Assume that NSP holds for some k and $\gamma < 1$
- For any $z, z' \in \mathcal{F}(y)$

$$\|z' - z\|_{\ell_1} \leq \frac{1+\gamma}{1-\gamma} \left(\|z'\|_{\ell_1} - \|z\|_{\ell_1} + 2\sigma_k(z)_{\ell_1} \right).$$

Proof of Geometric Property

- T the set of indices of the k largest entries in z

$$\begin{aligned}
\|(z' - z)_{T^c}\|_{\ell_1} &\leq \|z'_{T^c}\|_{\ell_1} + \|z_{T^c}\|_{\ell_1} \\
&= \|z'\|_{\ell_1} - \|z'_T\|_{\ell_1} + \sigma_k(z)_{\ell_1} \\
&= \|z\|_{\ell_1} + \|z'\|_{\ell_1} - \|z\|_{\ell_1} - \|z'_T\|_{\ell_1} + \sigma_k(z)_{\ell_1} \\
&\leq \|z_T\|_{\ell_1} - \|z'_T\|_{\ell_1} + \|z'\|_{\ell_1} - \|z\|_{\ell_1} + 2\sigma_k(z)_{\ell_1} \\
&\leq \|(z' - z)_T\|_{\ell_1} + \|z'\|_{\ell_1} - \|z\|_{\ell_1} + 2\sigma_k(z)_{\ell_1}
\end{aligned}$$

Proof of Geometric Property

- T the set of indices of the k largest entries in z

$$\begin{aligned}
\|(z'-z)_{T^c}\|_{\ell_1} &\leq \|z'_{T^c}\|_{\ell_1} + \|z_{T^c}\|_{\ell_1} \\
&= \|z'\|_{\ell_1} - \|z'_T\|_{\ell_1} + \sigma_k(z)_{\ell_1} \\
&= \|z\|_{\ell_1} + \|z'\|_{\ell_1} - \|z\|_{\ell_1} - \|z'_T\|_{\ell_1} + \sigma_k(z)_{\ell_1} \\
&\leq \|z_T\|_{\ell_1} - \|z'_T\|_{\ell_1} + \|z'\|_{\ell_1} - \|z\|_{\ell_1} + 2\sigma_k(z)_{\ell_1} \\
&\leq \|(z'-z)_T\|_{\ell_1} + \|z'\|_{\ell_1} - \|z\|_{\ell_1} + 2\sigma_k(z)_{\ell_1}
\end{aligned}$$

- Using NSP $\|(z'-z)_T\|_{\ell_1}$

$$\leq \gamma\|(z'-z)_{T^c}\|_{\ell_1} \leq \gamma(\|(z'-z)_T\|_{\ell_1} + \|z'\|_{\ell_1} - \|z\|_{\ell_1} + 2\sigma_k(z)_{\ell_1})$$

Proof Continued

- In other words,

$$\|(z' - z)_T\|_{\ell_1} \leq \frac{\gamma}{1-\gamma}(\|z'\|_{\ell_1} - \|z\|_{\ell_1} + 2\sigma_k(z)_{\ell_1}).$$

Proof Continued

- In other words,

$$\|(z' - z)_T\|_{\ell_1} \leq \frac{\gamma}{1-\gamma}(\|z'\|_{\ell_1} - \|z\|_{\ell_1} + 2\sigma_k(z)_{\ell_1}).$$

- Finally

$$\|z' - z\|_{\ell_1} = \|(z' - z)_{T^c}\|_{\ell_1} + \|(z' - z)_T\|_{\ell_1}$$

$$\leq \frac{1+\gamma}{1-\gamma}(\|z'\|_{\ell_1} - \|z\|_{\ell_1} + 2\sigma_k(z)_{\ell_1}),$$