Stochastic Galerkin approximations of elliptic PDEs driven by spatial white noise

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Two stochastic elliptic models

- Model I: $-\nabla \cdot ((\mathbf{E}[a](x) + \epsilon(x, \omega))\nabla u) = f$
  - $\epsilon(x, \omega)$ is a colored noise with a known correlation function.
- Model II: $-\nabla \cdot ((\mathbf{E}[a](x) + \dot{W}(x, \omega)) \diamond \nabla u) = f$
  - $\dot{W}(x, \omega)$ is spatial white noise on $L_2(D)$.
  - '$\diamond$' indicates the Wick product corresponding to Itô-Skorokhod integral.

Outline

- A brief overview of numerical methods for Model I
  - Karhunen-Loève expansion of the noise.
  - Polynomial chaos methods and variants.
- A stochastic finite element method for Model II
  - Spectral expansion of the white noise.
  - Weighted Wiener chaos space.
  - A stochastic FEM method.
- A simple comparison between Model I and II.

Karhunen-Loève expansion of the colored noise

Let $\epsilon(\boldsymbol{x},\omega)$ be a *second-order* random process with zero mean and unit variance, i.e., $\mathbb{E}[\epsilon](\boldsymbol{x})=0$, $\mathbb{E}[\epsilon^2](\boldsymbol{x})=1$. If the correlation function $R(\boldsymbol{x},\boldsymbol{y})=\mathbb{E}[\epsilon(\boldsymbol{x},\omega)\epsilon(\boldsymbol{y},\omega)]$ is known, the noise can be expressed as

$$\epsilon(x,\omega)=\sum_{i=1}^{\infty}\sqrt{\lambda_i}h_i(\boldsymbol{x})\xi_i$$

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$$\epsilon(x,\omega) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} h_i(\boldsymbol{x}) \xi_i$$

- $\{(\sqrt{\lambda_i}, h_i(\boldsymbol{x}))\}_{i=1}^{\infty}$ are eigen-pairs of $R(\boldsymbol{x},\boldsymbol{y})$, where

$$\int_D R(\boldsymbol{x},\boldsymbol{y}) h_i(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} = \lambda_i h_i(\boldsymbol{x}), \quad \int_D h_i(\boldsymbol{x}) h_j(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \delta_{ij}.$$

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- $\{\xi_i\}$ is a set of *mutually uncorrelated* random variables with zero mean and unit variance.
- The convergence of K-L expansion is optimal in the L_2 sense.

Approximate the problem using a series of random variables

$$\begin{cases} -\nabla \cdot (a(\boldsymbol{x}, \omega)\nabla u(\boldsymbol{x}, \omega)) = f(\boldsymbol{x}) \text{ on } D, \\ u(\boldsymbol{x}, \omega) = 0 \text{ on } \partial D. \end{cases}$$

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$$\begin{cases} -\nabla \cdot (a_M(\boldsymbol{x},\boldsymbol{\xi})\nabla u(\boldsymbol{x},\boldsymbol{\xi})) = f(\boldsymbol{x}) \text{ on } D, \\ u(\boldsymbol{x},\boldsymbol{\xi}) = 0 \text{ on } \partial D. \end{cases}$$

- $a_M(\boldsymbol{x},\xi) = \mathbb{E}[a](\boldsymbol{x}) + \sigma \sum_{i=1}^{M} \sqrt{\lambda_i} h_i(\boldsymbol{x}) \xi_i$.
- $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_M)$, and σ indicates the degree of perturbation.

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Need to approximate the random function $u(\boldsymbol{x}, \boldsymbol{\xi})$ efficiently!

(Generalized) polynomial chaos (gPC)

- Polynomial chaos: [Wiener, 38; Cameron and Martin, 47]

  Let C be the space of continuous functions induced by the Wiener process $\{W_t, 0 < t < 1\}$. If F is a functional of $L_2(C)$, i.e., $\mathbb{E}[F^2] < \infty$, then the Fourier-Hermite expansion

$$F = \sum_{|\alpha|=0}^{\infty} f_\alpha \phi_\alpha(\boldsymbol{\xi}) = \sum_{|\alpha|=0}^{\infty} f_\alpha \prod_{j=1}^{\infty} H_{j,\alpha_j}(\xi_j), \quad \xi_j = \int_0^1 b_j(t) \mathrm{d}W_t$$

  converges in the $L_2(C)$ sense, where $\alpha_i \in \mathbb{N}_0$, $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$, $\{b_i(t)\}$ is a complete orthonormal set of real functions in $L_2(0,1)$.

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(Generalized) polynomial chaos (gPC) - cont'd.

- Generalized polynomial chaos: [Xiu and Karniadakis, 02]

$$u(\boldsymbol{x};\boldsymbol{\xi}) = \sum_{|\alpha|=0}^{\infty} u_\alpha(\boldsymbol{x}) \boxed{\phi_\alpha(\boldsymbol{\xi})} = \sum_{|\alpha|=0}^{\infty} u_\alpha(\boldsymbol{x}) \boxed{\prod_{j=1}^{M} \phi_{j,\alpha_j}(\xi_j)},$$

- $\alpha \in \mathbb{N}_0^M$, $\boldsymbol{\xi} \in \mathbb{R}^M$, $|\alpha| = \sum_{i=1}^M \alpha_i$.
- ξ_i are *independent*; $\mathbb{E}[\phi_\alpha \phi_\beta] = \delta_{\alpha\beta}$ w.r.t. the PDF $f(\boldsymbol{\xi})$
- Correspondence between PDF and classical orthogonal polynomials: uniform - Legendre; Gaussian -Hermite, etc.
- L_2 completeness of orthogonal polynomials.

Apply gPC to the stochastic elliptic problem

$$L(\boldsymbol{x}, u; \boldsymbol{\xi}) = f(\boldsymbol{x})$$

- Galerkin projection:
  - PC expansions: $u = \sum_{|\alpha|=0}^{p} u_\alpha \phi_\alpha$.
  - Residual: $R(\boldsymbol{\xi}) = L(\boldsymbol{x}, \sum_{|\alpha|=0}^{p} u_\alpha \phi_\alpha) - f(\boldsymbol{x})$.
  - Deterministic system of u_α: $\boxed{\mathbb{E}\left[R(\boldsymbol{\xi})\phi_\beta(\boldsymbol{\xi})\right] = 0, \quad |\beta| = 0, \ldots, p.}$
  - # of ϕ_α: $\frac{(M+p)!}{M!p!}$

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- Collocation projection:
  - Interpolation operator: $\{\boldsymbol{\xi}^{(j)}\}_{j=1}^{N_g}$: a set of grid points in the parametric space.
  - Deterministic system on grid points: $\boxed{L(\boldsymbol{x}, u; \boldsymbol{\xi}^{(j)}) = f(\boldsymbol{x}).}$
  - Choices of $\{\boldsymbol{\xi}^{(j)}\}$: full tensor-products of Gauss quadrature points - $O(N^M)$, sparse grids - $O(N \log(N)^{M-1})$

Comments on (generalized) polynomial chaos

- Advantages of gPC:
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- Advantages of gPC:
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  - Efficiency due to orthogonality.

- Disadvantages of gPC:
  - Efficency decreases as the number of random dimensions increases.
  - Inefficient for problems with low regularity in the parametric space.
  - *May diverge for long-time integrations.*

Variants of the polynomial chaos method

- Choices of global approximation bases.
  - Sparse polynomial chaos bases. [Schwab et al., Webster et al.]

$$V := \{\phi_\alpha \mid |\alpha| \leq p\}$$

Error

$$V_s := \{\phi_\alpha \mid \alpha_i \leq \beta_i\}$$

$$\longrightarrow \quad \dim(V_s) < \dim(V)$$

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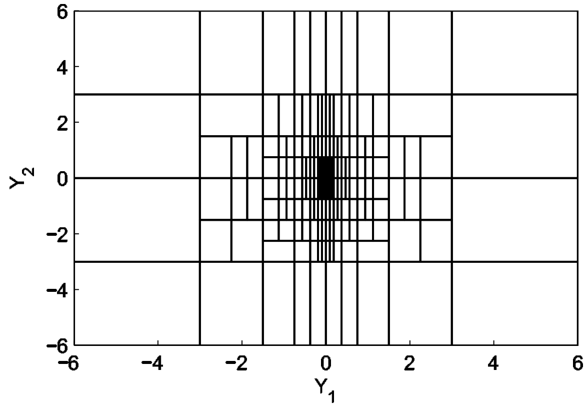
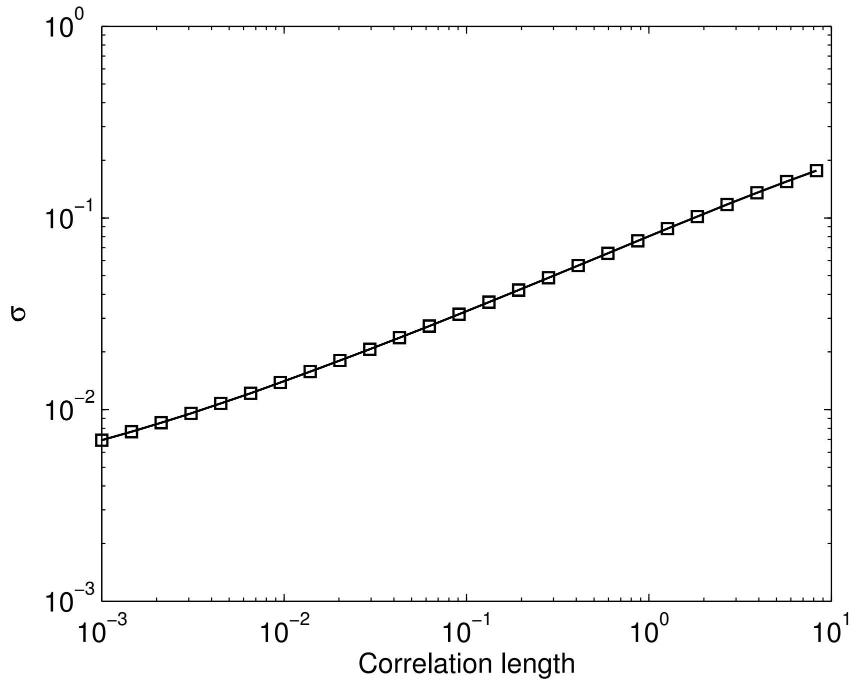


- Choices of local approximation bases - $\xi \in \Gamma$
  - Piecewise finite element space. [Babuska et al.]
  - Wavelets approximation. [Le Maitre et al.]
  - Adaptive multi-element gPC. [Wan and Karniadakis]

Strong ellipticity and Model I

Strong ellipticity: $a_M = \mathbb{E}[a](\boldsymbol{x}) + \sum_{i=1}^{M} \sqrt{\lambda_i} h_i(\boldsymbol{x}) \xi_i > c > 0$ a.s.



As $M \to \infty$, strong ellipticity condition may fail.

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Stochastic elliptic PDE - Model II (white noise)

$$\begin{cases} \mathbf{A}u + \boxed{\delta(\mathbf{M}u)} = f(\boldsymbol{x}) & \text{on } D, \\ u(\boldsymbol{x}) = 0 & \text{on } \partial D. \end{cases}$$

$$\begin{aligned} \mathbf{A}u(\boldsymbol{x}) &:= -D_i(a_{ij}(\boldsymbol{x})D_j u(\boldsymbol{x})) \\ \delta(\mathbf{M}u) &= \text{Itô-Skorokhod integral of } \mathbf{M}u \\ \mathbf{M}u(\boldsymbol{x}) &:= D_i(\sigma_{ij}(\boldsymbol{x})D_j u(\boldsymbol{x})) \end{aligned}$$

Note: $\mathbf{M}$ *can be an operator up to order two.*

White noise on a real separable Hilbert space $\mathcal{U}$

Given a complete orthonormal basis $\{w_k\}_{k=1}^{\infty}$ in $\mathcal{U}$ and a zero-mean Gaussian family $\dot{W} = \{\dot{W}(h),\, h \in \mathcal{U}\}$ such that

$$\mathbb{E}[\dot{W}(h_1)\dot{W}(h_2)] = (h_1, h_2)_{\mathcal{U}}, \quad \forall h_1, h_2 \in \mathcal{U},$$

the formal series

$$\dot{W} = \sum_{k=1}^{\infty} \dot{W}(w_k) w_k$$

is called *(Gaussian) white noise* on $\mathcal{U}$.

Note: Due to the fact that $(w_i, w_j) = \delta_{ij}$, $\dot{W}(w_k) \sim \mathcal{N}(0, 1)$.

Spectral expansion of $\dot{W}$: $\dot{W} = \sum_{k \geq 1} w_k(\boldsymbol{x})\xi_k$, $\xi_k \sim \mathcal{N}(0, 1)$.

Weighted Wiener-chaos space

- Define a linear bounded operator $\mathcal{R}$ on $L_2(\mathbb{F} = (\Omega, \mathcal{F}, \mathbb{P}))$:

$$\mathcal{R} H_\alpha = r_\alpha H_\alpha, \quad \mathcal{R}^{-1} H_\alpha = r_\alpha^{-1} H_\alpha, \quad 0 < r_\alpha < C.$$

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- $\mathcal{R} L_2(\mathbb{F}; X)$: the closure of $L_2(\mathbb{F}; X)$ with respect to the norm

$$\|f\|^2_{\mathcal{R}L_2(\mathbb{F};X)} := \|\mathcal{R}f\|^2_{L_2(\mathbb{F};X)} = \sum_{\alpha \in \mathcal{J}} \|f_\alpha\|^2_X \alpha! r_\alpha^2,$$

where the chaos expansion of f takes the form $f = \sum_{|\alpha|=0}^{\infty} f_\alpha H_\alpha$.

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- Given $f \in \mathcal{R}L_2(\mathbb{F}; X)$ and $g \in \mathcal{R}^{-1} L_2(\mathbb{F}; X)$, we define a scalar inner product as

$$\langle\langle f, g \rangle\rangle := \mathbb{E}[(\mathcal{R}f, \mathcal{R}^{-1} g)_X].$$

Wick product and Itô-Skorokhod integral

- Wick product with respect to Hermite polynomials $H_\alpha(\boldsymbol{\xi})$:
  - $H_\alpha(\boldsymbol{\xi}) \diamond H_\beta(\boldsymbol{\xi}) = H_{\alpha+\beta}(\boldsymbol{\xi}), \quad \forall \alpha, \beta \in \mathbb{N}_0^{\mathbb{N}}$

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- Wick product and Itô-Skorokhod integral - an example

$$\int_{[0,T]} f \mathrm{d}W_t = \int_{[0,T]} f \diamond \dot{W} \mathrm{d}t,$$

where W_t is a one-dimensional Wiener process.

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- Itô-Skorokhod integral of $\mathbf{M}u$:

$$\delta(\mathbf{M}u) = \mathbf{M}u \diamond \dot{W} := \sum_{k \geq 1} \mathbf{M}_k u \diamond \xi_k,$$

where $\mathbf{M}_k u(\boldsymbol{x}) := w_k(\boldsymbol{x}) D_i(\sigma_{ij}(\boldsymbol{x}) D_j u(\boldsymbol{x}))$.

Stochastic elliptic PDE - Model II (white noise)

$$\begin{cases} \mathbf{A}u + \ \delta(\mathbf{M}u) & = & f(\boldsymbol{x}) & \text{on } D, \\ u(\boldsymbol{x}) & = & 0 & \text{on } \partial D, \end{cases}$$

$$\dot{W}(\boldsymbol{x}) = \sum_{k=1}^{\infty} w_k(\boldsymbol{x})\xi_k, \qquad \delta(\mathbf{M}u) = \mathbf{M}u \diamond \dot{W} := \sum_{k\geq 1} \mathbf{M}_k u \diamond \xi_k$$

$$\mathbf{A}u(\boldsymbol{x}) := -D_i(a_{ij}(\boldsymbol{x})D_j u(\boldsymbol{x})), \quad \mathbf{M}_k u(\boldsymbol{x}) := w_k(\boldsymbol{x})D_i(\sigma_{ij}(\boldsymbol{x})D_j u(\boldsymbol{x}))$$

Assumption:

- $a_{ij}(\boldsymbol{x})$ and $\sigma_{ij}(\boldsymbol{x})$ are measurable and bounded in $\bar{D}$.
- $A_1|\boldsymbol{y}|^2 \leq a_{ij}y_iy_j \leq A_2|\boldsymbol{y}|^2, \forall \boldsymbol{x} \in \bar{D}, \boldsymbol{y} \in \mathbb{R}^d$, where A_1 and A_2 are positive numbers.
- w_k is bounded and Lipschitz continuous.

Model II is unbiased

$$u = \sum_{\alpha \in \mathbb{N}_0^{\mathbb{N}}} u_\alpha H_\alpha \Rightarrow \mathbb{E}[\delta(\mathbf{M}u)] = \sum_{k \geq 1} \sum_{\alpha \in \mathbb{N}_0^{\mathbb{N}}} \mathbf{M}_k u_\alpha \mathbb{E}[H_\alpha \diamond \xi_k] = 0$$

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$$\mathbb{E}[\mathbf{A}u + \delta(\mathbf{M}u)] = \mathbb{E}[f]$$

$$\downarrow$$

$$\mathbf{A}\mathbb{E}[u] = \mathbb{E}[f]$$

Mean of the Stochastic PDE is the unperturbed deterministic PDE.

A variational approach

- Function spaces:

$$V := \mathcal{R}L_2(\mathbb{F}; H_0^1(D)) \quad \hat{V} := \mathcal{R}^{-1}L_2(\mathbb{F}; H_0^1(D)).$$

- Bilinear form:

$$\mathscr{B}(u,v) = \langle\langle \mathbf{A}u, v\rangle\rangle + \langle\langle \mathbf{M}u \diamond \dot{W}, v\rangle\rangle.$$

- Linear form:

$$\mathscr{L}(v) = \langle\langle f, v\rangle\rangle.$$

Find a solution $u \in V$ satisfying the weak form:

$$\mathscr{B}(u,v) = \mathscr{L}(v), \quad \forall v \in \hat{V}.$$

Uncertainty propagator

- Let
$$u(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{|\alpha|=0}^{\infty} u_\alpha h_\alpha(\boldsymbol{\xi}),$$
where $h_\alpha(\boldsymbol{\xi})$ are normalized Hermite polynomials.

- The Galerkin projection in probability space yields ‘uncertainty propagator’ as
$$\mathbf{A}u_\alpha + \sum_{k\geq 1} \sqrt{\alpha_k}\mathbf{M}_k u_{\alpha-\epsilon_k} = f_\alpha,$$
where $\epsilon_k = (0,\ldots,1,\ldots,0)$: only the k-th component equal to 1.
    - The ellipticity of $\mathbf{A}$ guarantees the wellposedness.
    - u_α only depends on coefficients $\beta < \alpha$.

Equivalence between the model problem and its propagator

Theorem. *There exist a unique solution $u \in \mathcal{R}L_2(\mathbb{F}; H_0^1(D))$, if the operator $\mathcal{R}$ is defined by the weights r_α as*

$$r_\alpha = \frac{q^\alpha}{2^{|\alpha|}\sqrt{|\alpha|!}}, \quad \textit{with} \quad q^\alpha = \prod_{k=1}^{\infty} q_k^{\alpha_k},$$

where the number q_k are chosen so that

$$\boxed{\sum_{k\geq 1} k^2 q_k^2 C_k^2 < 1.}$$

$$\begin{aligned} \|u_0\|_{H_0^1(D)} &\leq C_A \|f\|_{H^{-1}(D)}, \\ \|\mathbf{A}^{-1}\mathbf{M}_k v\|_{H_0^1(D)} &\leq C_k \|v\|_{H_0^1(D)}, \quad \forall v \in H_0^1(D). \end{aligned}$$

Continuity of $\mathbf{A}^{-1}\mathbf{M}_k$

- $|w_k(\boldsymbol{x})| \le C_k^w$ and $|w_k(\boldsymbol{x}) - w_k(\boldsymbol{y})| \le C_k^L |\boldsymbol{x} - \boldsymbol{y}|, \ \forall \boldsymbol{x}, \boldsymbol{y} \in \bar{D}$.

$$
\begin{aligned}
A_1 \|\hat{v}\|^2_{H_0^1(D)} &\le (\mathbf{A}\hat{v}, \hat{v}) = (\mathbf{M}_k v, \hat{v}) \\
&= \textstyle\sum_{i,j} (w_k D_i(\sigma_{ij} D_j v), \hat{v}) = \sum_{i,j} (\sigma_{ij} D_j v, D_i(w_k \hat{v})) \\
&= \textstyle\sum_{i,j} [(\sigma_{ij} D_j v, \hat{v} D_i w_k) + (\sigma_{ij} D_j v, w_k D_i \hat{v})] \\
&\le \max_{i,j} \sigma_{ij} [C_k^L \|v\|_{H_0^1(D)} \|\hat{v}\|_{L_2(D)} + C_k^w \|v\|_{H_0^1(D)} \|\hat{v}\|_{H_0^1(D)}] \\
&\le \max_{i,j} \sigma_{ij} (C_k^L C_p + C_k^w) \|v\|_{H_0^1(D)} \|\hat{v}\|_{H_0^1(D)},
\end{aligned}
$$

Thus,

$$
\|\hat{v}\|_{H_0^1(D)} = \|\mathbf{A}^{-1}\mathbf{M}_k v\|_{H_0^1(D)} \le \max_{i,j} \sigma_{ij} (C_k^L C_p + C_k^w)/A_1 \|v\|_{H_0^1(D)},
$$

where C_p is the Poincaré constant in D.

A stochastic FEM method

- $V_h \subset H_0^1(D)$: finite element space for the physical discretization.
- Truncated Wiener chaos space:

$$V_c := \{f = \sum_{\alpha \in \mathbb{N}_0^M, |\alpha| \leq p} f_\alpha h_\alpha | f_\alpha \in \mathbb{R}, \, \|f\|_{\mathcal{R}L_2(\mathbb{F})} < \infty\}$$

- The dual space of V_c:

$$V_c^{-1} := \{f = \sum_{\alpha \in \mathbb{N}_0^M, |\alpha| \leq p} f_\alpha h_\alpha | f_\alpha \in \mathbb{R}, \, \|f\|_{\mathcal{R}^{-1}L_2(\mathbb{F})} < \infty\}$$

Find a solution $u_h^{M,p} \in V_h \otimes V_c$ such that

$$\langle\langle \mathbf{A} u_h^{M,p}, v \rangle\rangle + \langle\langle \textstyle\sum_{k=1}^M \mathbf{M}_k u_h^{M,p} \diamond \xi_k, v \rangle\rangle = \langle\langle f, v \rangle\rangle, \; \forall v \in V_h \otimes V_c^{-1}.$$

Convergence of the stochastic FEM

Theorem. *Assume that* $\alpha_{ij}, \sigma_{ij}, w_k$ *have proper regularity and* $u \in \mathcal{R}L_2(\mathbb{F}; H) \cap \mathcal{R}L_2(\mathbb{F}; H^{m+1}(D))$. *The approximation solution* $u_h^{M,p}$ *given by the stochastic finite element method can be bounded as*

$$\|u - u_h^{M,p}\|_{\mathcal{R}L_2(\mathbb{F};H)} \leq C\left(\boxed{h^m \|u\|_{\mathcal{R}L_2(\mathbb{F};H^{m+1}(D))}} + \boxed{\frac{\hat{q}_W}{1-\hat{q}}} + \boxed{\frac{\hat{q}^{p+1}}{1-\hat{q}}} \right),$$

where the constant C *is independent of* h.

$$\hat{q}_W = \sum_{k>M} k^2 C_k^2 q_k^2 \text{ and } \hat{q} = \sum_{k\geq 1} k^2 C_k^2 q_k^2 < 1.$$

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Error from the finite element discretization int the physical space.

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$\hat{q}_W = \sum_{k>M} k^2 C_k^2 q_k^2$ *and* $\hat{q} = \sum_{k\ge 1} k^2 C_k^2 q_k^2 < 1.$

Error from the truncation in approximation of white noise:

$\hat{q}_W = \sum_{k>M} k^2 C_k^2 q_k^2$ as $M \to \infty$.

Convergence of the stochastic FEM

Theorem. *Assume that* $\alpha_{ij}, \sigma_{ij}, w_k$ *have proper regularity and* $u \in \mathcal{R}L_2(\mathbb{F}; H) \cap \mathcal{R}L_2(\mathbb{F}; H^{m+1}(D))$. *The approximation solution* $u_h^{M,p}$ *given by the stochastic finite element method can be bounded as*

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Error from the truncation of Wiener-chaos expansion:

spectral convergence with respect to the weighted norm.

An numerical example

$$\begin{cases} -\nabla \cdot [(\mathbb{E}[a](\boldsymbol{x}) + \epsilon \dot{W}) \diamond \nabla u(\boldsymbol{x}, \omega)] = f(\boldsymbol{x}), \ \boldsymbol{x} \in D, \\ u(\boldsymbol{x}, \omega) = 0, \ \boldsymbol{x} \in \partial D, \end{cases}$$

- $D = (0,1)^2$, $\mathbb{E}[a](\boldsymbol{x}) = 1$, $\epsilon = 1$, and $f(\boldsymbol{x}) = 1$.
- Orthonormal basis on $L_2(D)$:

$$w_{m,n}(\boldsymbol{x}) = \begin{cases} 1, & m = n = 0 \\ \sqrt{2}\sin(m\pi x), & n = 0 \\ \sqrt{2}\sin(n\pi y), & m = 0 \\ 2\sin(m\pi x)\sin(n\pi y), & m, n = 1, 2, \ldots, \infty. \end{cases}$$

- $\mathbf{M}_k u = -\nabla \cdot (w_k(\boldsymbol{x}) \nabla u)$ and continuity of $\mathbf{A}^{-1}\mathbf{M}_k$:

$$C_k = \max w_k(\boldsymbol{x}) \leq \hat{C}_k = \begin{cases} 1, & \text{m=n=0}, \\ \sqrt{2}, & mn = 0, m + n > 0, \\ 2, & \text{otherwise}. \end{cases}$$

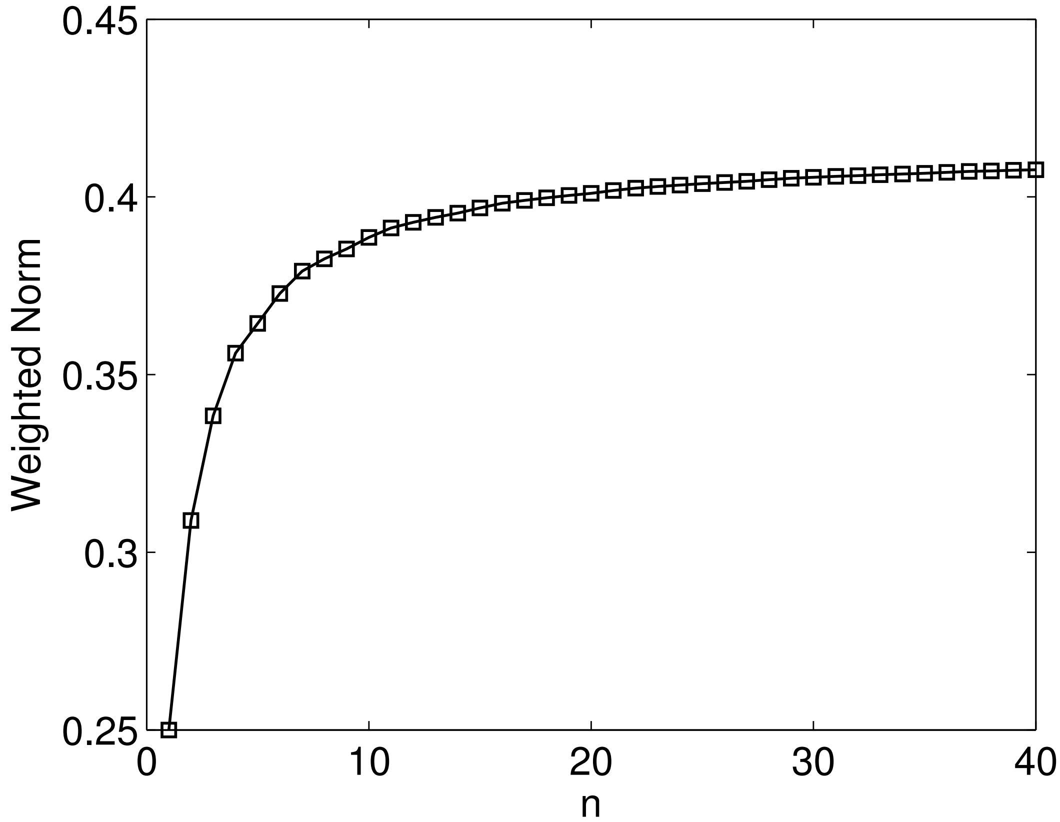
Convergence of approximation of white noise

Weights r_α :

$$r_\alpha = \frac{q^\alpha}{2^{|\alpha|}\sqrt{|\alpha|!}}$$

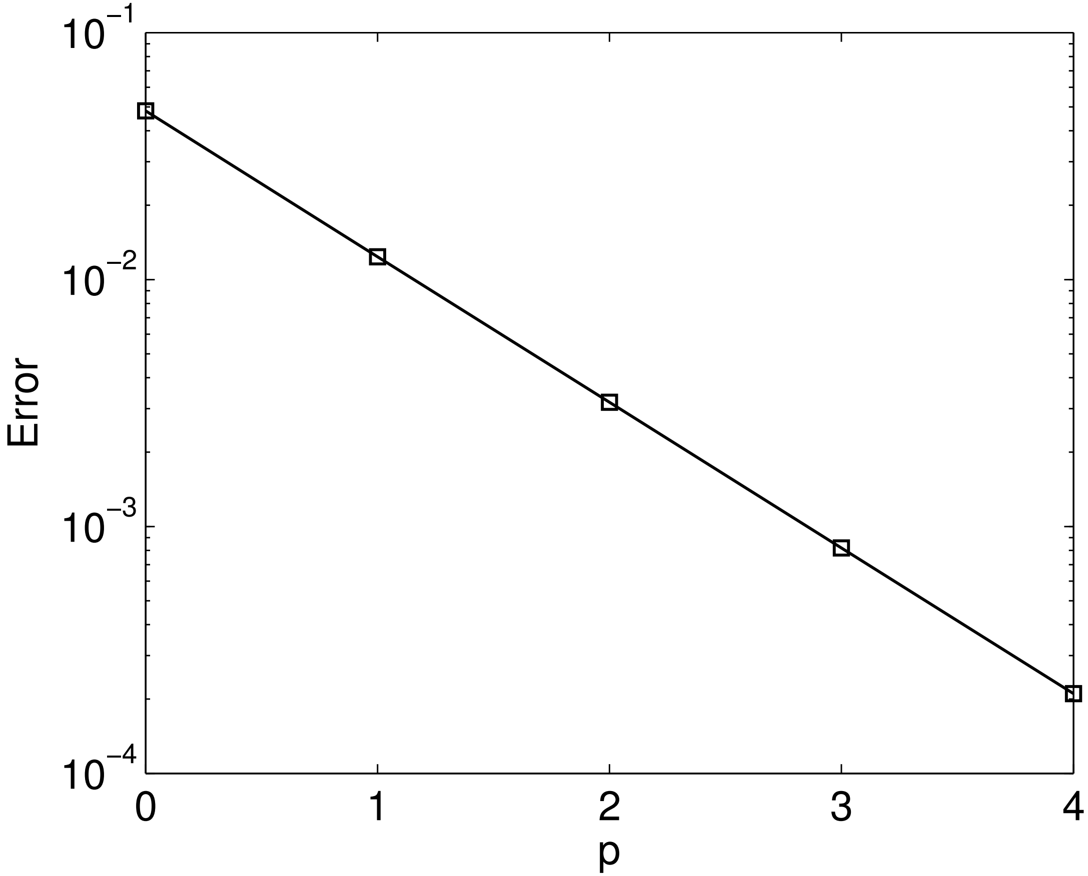
$$q^\alpha = \prod_{k=1}^{\infty} q_k^{\alpha_k}$$

$$q^k = \frac{1}{(k+1)k\hat{C}_k}$$



Weighted L_2 norm of approximate white noise.

p-convergence of the approximate solution



Spectral convergence of $\|u_h^{M,p}\|_{\mathcal{R}L_2(\mathbb{F};H_0^1(D))}$. $M = 21$.

Generalization of Model II

Itô-Skorokhod integral: $\delta(\mathbf{M}u) = \mathbf{M}u \diamond \dot{W}$

↓

Convolution with respect to $g \in \mathcal{R}L_2(\mathbb{F}; X)$:

$$\mathbf{M}u \diamond g = \mathbf{M}u \diamond \sum_{\alpha \in \mathbb{N}_0^{\mathbb{N}}} g_\alpha H_\alpha$$

Comparison between Wick and ordinary products

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}x}\left(K(x,\omega) * \frac{\mathrm{d}u}{\mathrm{d}x}\right) = 1 \\ u(0) = 0, \quad u(1) = 0. \end{cases}$$

- '$*$' indicates Wick product '$\diamond$' or ordinary product '$\cdot$'.
- $K(x) = \mathrm{e}^{c\xi - \frac{1}{2}c^2}$, $\xi \sim \mathcal{N}(0,1)$, c is constant.
- $\mathbb{E}[K] = 1$, Var$[K] = \mathrm{e}^{c^2} - 1$.

Wellposedness of the 1D models

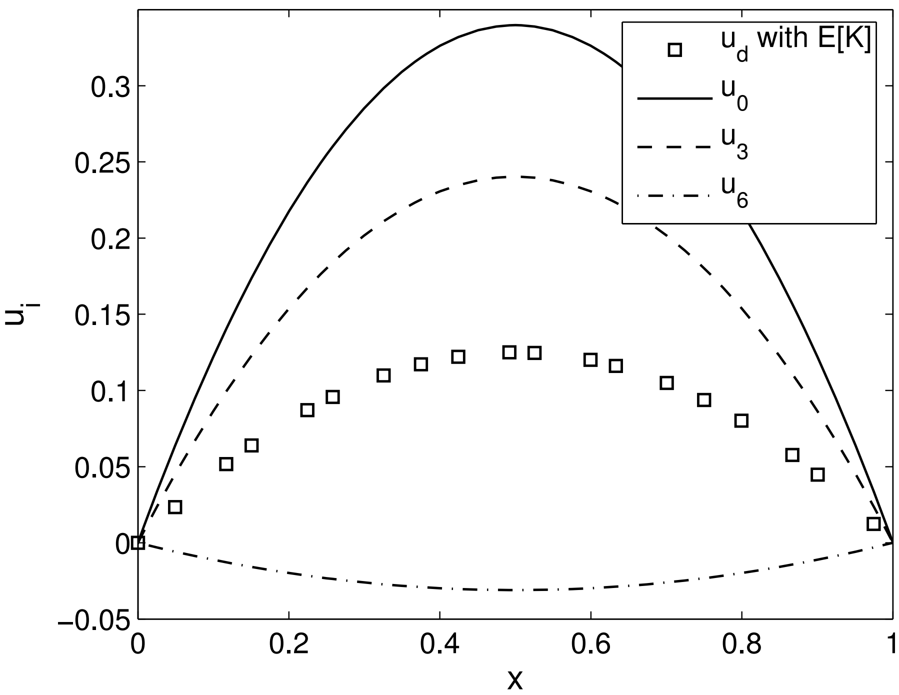
$$u(x) = \sum_{i=0}^{p} u_i(x) h_i(\xi)$$

$$\begin{cases} -\sum_{i=0}^{p} \mathbb{E}[K\xi_i\xi_j]\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_i & = & \delta_{0j}, \forall j = 0, \dots, p \\ -\sum_{i=0}^{p} \mathbb{E}[(K \diamond \xi_i)\xi_j]\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_i & = & \delta_{0j}, \forall j = 0, \dots, p \end{cases}$$

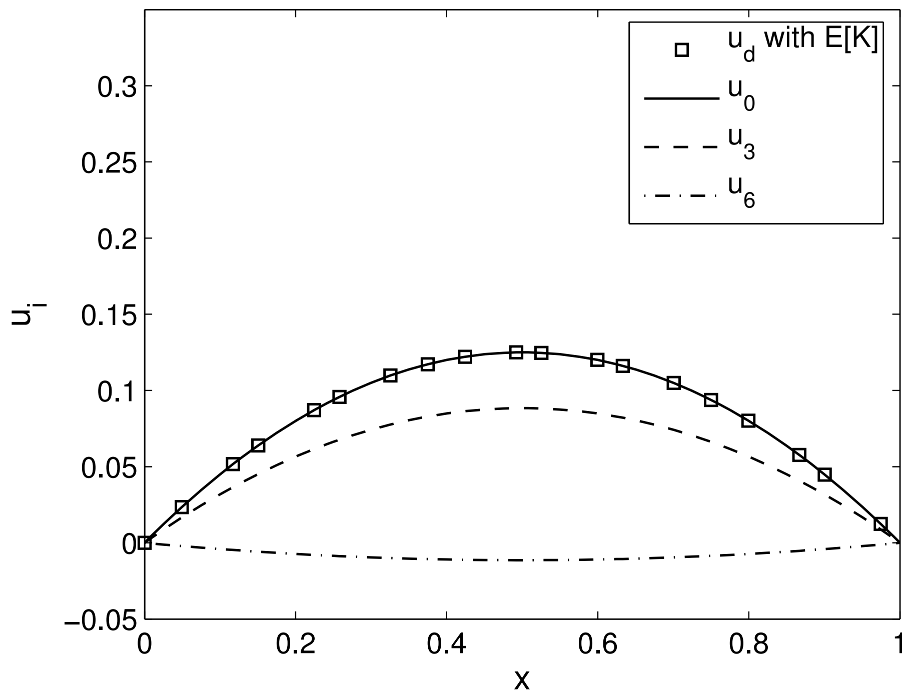
- $\mathbb{E}[K\xi_i\xi_j]$ is symmetric and nonnegative-definite.
- $\mathbb{E}[(K \diamond \xi_i)\xi_j]$ is lower-triangular with $\mathbb{E}[(K \diamond \xi_i)\xi_i] = 1$, which means that u_i only depends on u_j with $j < i$.

Both systems are wellposed.

Comparison between coefficients of Wiener chaos expansions

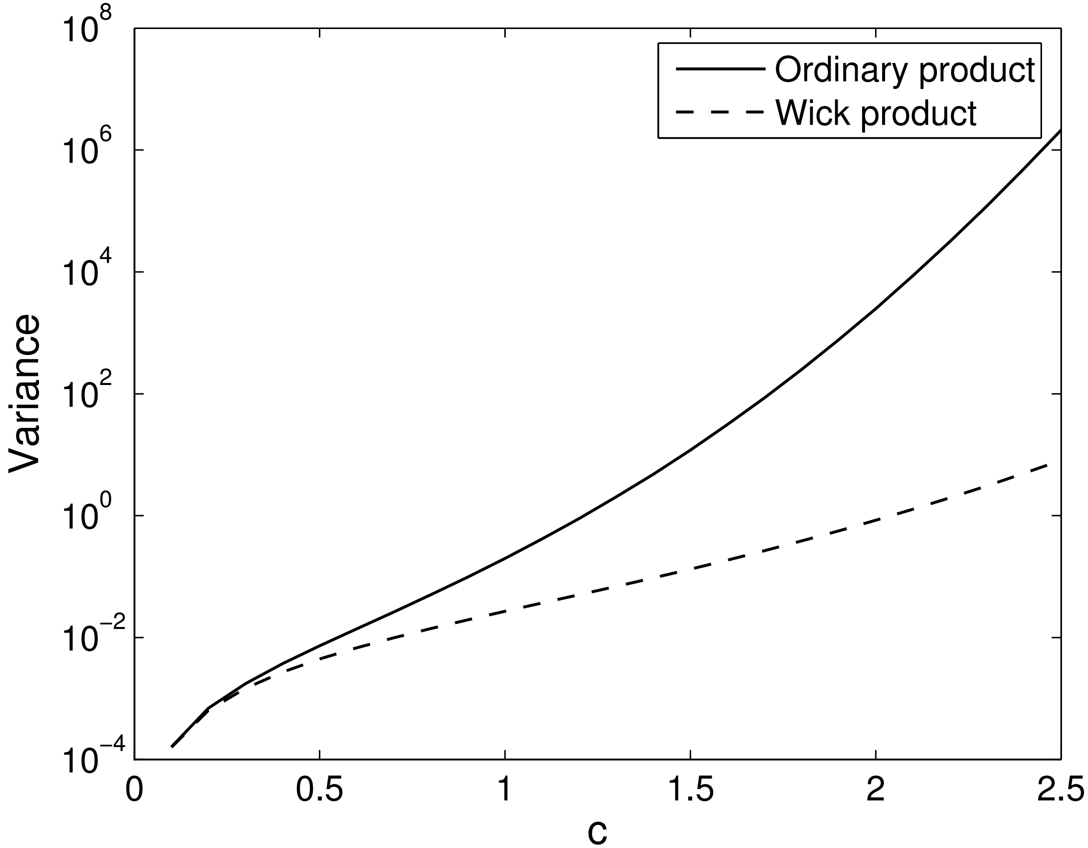


Ordinary product



Wick product

Comparison between variances



Variance versus perturbation at $x = 0.5$.

Summary

- A brief overview of polynomial chaos methods for elliptic PDEs perturbed by colored noise.
  - Karhunen-Loève expansion of colored noise.
  - Adaptive polynomial chaos methods.
- A stochastic FE method for elliptic PDEs perturbed by spatial white noise.
  - Itô-Skorokhod integral - convolution with respect to white noise through the wick product.
  - Weighted Wiener chaos space
  - p-convergence under the weighted norm